On algebraic properties of bicomplex and hyperbolic numbers.

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Abstract

In this article we present, in a unified manner, a variety of algebraic properties of both bicomplex numbers and hyperbolic numbers. In particular, using three types of conjugations, we describe in detail some specific moduli with complex and hyperbolic ranges. We also specify for hyperbolic numbers all the properties already established for bicomplex numbers. Finally we look at these algebraic structures in the specific context of Clifford Algebras.

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Introduction

The aim of this work is to present in a unified manner a variety of algebraic properties of both bicomplex numbers and hyperbolic numbers, the latter seen as a particular and important case of the former. Of course, the topic itself is not a novelty. In 1892, in search for special algebras, Corrado Segre (1860-1924) published a paper $[Seg]$ in which he treated an algebra whose elements are called bicomplex numbers. In fact, many properties of such numbers are dispersed over a number of books and articles; we can recommend the books $[BP], [El], [Ol]$

and especially $[Ga]$ containing a good list of references also which would allow an interested reader to deepen his familiarity with both types of numbers; the books $[Ya1]$ and $[Ya2]$ could be helpful as well.
Mostly the authors deal with the analysis of bicomplex and hyperbolic functions thus mentioning only the necessary algebraic backgrounds; see for instance, the following brief and rather random extraction from the long list of possible references: $[MaKaMe], [MoRo], [Shp1], [Shp2], [Ry1], [Ry2], [Xu]$. Sometimes bicomplex numbers appear quite suddenly in a context which, at first sight, has nothing to do with them; as a illustration we note the book $[KrSh]$ where on pp. $95-111$ one can see how useful they are in describing important properties of hyperholomorphic functions with values in complex quaternions.
In a sense, the above fact together with $[Ro1], [Ro2]$ gave rise to the desire to have an ample and systematic exposition of algebraic aspects of the theory of bicomplex numbers.
The paper is organized as follows. Section 1 introduces basic definitions and notations followed, in Section 2, by different types of conjugations generalizing the usual complex one. The three types of the bicomplex conjugation lead to different moduli of bicomplex numbers described in Section 3 and 4. Although the hyperbolic numbers are introduced at the very beginning, in Section 1, we specify for them all the properties already established for bicomplex numbers, in Section 5. In the last Section 6 both sets, of bicomplex and of hyperbolic numbers, are looked at from the setting of Clifford Algebras, and their special place among them is explained.

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1 Bicomplex numbers: basic definitions.

1.1

We assume reader's acquaintance with the theory of real and of complex numbers. In particular, we assume that the reader possesses the idea that the set of complex numbers, $\mathbb{C}$, can be viewed as an algebra generated by the field of real numbers, $\mathbb{R}$, and by a new, non-real element $\mathbf{i}$ whose main property is $\mathbf{i}^2 = -1$. All this means that

$$\mathbb{C} := \{x + y\mathbf{i} \mid x, y \in \mathbb{R}\}.$$

Thus, the set $\mathbb{C}$ is a kind of a "duplication" of the real numbers.

1.2

What we want to do now is to try to repeat this duplicating process, but now to apply it to the set of complex numbers. For doing this, let us denote the imaginary unit in $\mathbb{C}$ by $\mathbf{i_1}$ thus writing also $\mathbb{C}(\mathbf{i_1}) := \{x + y\mathbf{i_1} \mid \mathbf{i_1}^2 = -1\}$.

Let, then, $\mathbf{i_2}$ denote a new element (the second imaginary unit) with the properties:

$$\mathbf{i_2}^2 = -1; \quad \mathbf{i_1}\mathbf{i_2} = \mathbf{i_2}\mathbf{i_1}; \quad \alpha\mathbf{i_2} = \mathbf{i_2}\alpha \quad \forall\alpha \in \mathbb{R}.$$

This means, in particular, that there have been introduced a commutative multiplication of the elements $\mathbf{i_1}$ and $\mathbf{i_2}$, and we can extend it now onto the whole set

$$\mathbb{T} := \mathbb{C}_2 := \{z_1 + z_2\mathbf{i_2} \mid z_1, z_2 \in \mathbb{C}(\mathbf{i_1})\}$$

to an associative, distributive, and commutative multiplication by the rule:

$$(z_1 + z_2\mathbf{i_2})(z_3 + z_4\mathbf{i_2}) := (z_1z_3 - z_2z_4) + (z_1z_4 + z_2z_3)\mathbf{i_2}.$$

Of course, being formally a definition, it is a result of an algebraic manipulation, as well as the following rules:

$$z_1 + z_2\mathbf{i_2} = z_3 + z_4\mathbf{i_2} \iff z_1 = z_3 \text{ and } z_2 = z_4,$$

$$(z_1 + z_2\mathbf{i_2}) + (z_3 + z_4\mathbf{i_2}) := (z_1 + z_3) + (z_2 + z_4)\mathbf{i_2}.$$

We shall call the elements of the set $\mathbb{C}_2 = \mathbb{T}$ *bicomplex numbers* by obvious reasons. The “real” and “complex” zero 0 is identified with the “bicomplex zero” $0 + 0\mathbf{i_2}$, as well as the “real” and “complex” unit 1 coincides with the bicomplex unit $1 + 0\mathbf{i_2}$.

1.3

It is easy to verify that with these operations $\mathbb{T}$ becomes a commutative ring with unit. Identifying $\mathbb{R}$ with the set

$$\{z_1 + z_2\mathbf{i_2} \mid z_2 = 0, z_1 \in \mathbb{R}\} \subset \mathbb{T},$$

we make $\mathbb{R}$ a subring of $\mathbb{T}$. There is an obvious way of imbedding $\mathbb{C}$ into $\mathbb{T}$ by identifying it with the set

$$\mathbb{C}(\mathbf{i_1}) := \{z_1 + 0\mathbf{i_2}\} \subset \mathbb{T},$$

hence $\mathbb{C}$ becomes also a subring of $\mathbb{T}$. But one should be careful here since this imbedding is not the only possible one; indeed, we can identify $\mathbb{C}$ with the following set as well:

$$\mathbb{C}(\mathbf{i_2}) := \{z_1 + z_2\mathbf{i_2} \mid z_1, z_2 \in \mathbb{R}\}.$$

Although, of course, each of those subsets of $\mathbb{T}$, both $\mathbb{C}(\mathbf{i_1})$ and $\mathbb{C}(\mathbf{i_2})$, is isomorphic to $\mathbb{C}$ but seen inside $\mathbb{T}$, they are essentially different.

There are other interesting subsets in $\mathbb{T}$ which will be commented later on.

Note finally that introducing the set $\mathbb{T} = \mathbb{C}_2$ we have converted the complex linear space

$$\mathbb{C}^2 := \{(z_1, z_2) \mid z_1, z_2 \in \mathbb{C}\} = \mathbb{C} \times \mathbb{C}$$

into a complex (over $\mathbb{C}(\mathbf{i_1})$ or over $\mathbb{C}(\mathbf{i_2})$!) commutative algebra.

1.4

The bicomplex numbers have been introduced as pairs of usual complex numbers, with an additional structure of a commutative multiplication. Let $w = z_1 + z_2\mathbf{i_2}$, and let us write down the complex numbers z_1 and z_2 as $z_1 = w_1 + w_2\mathbf{i_1}$, $z_2 = w_3 + w_4\mathbf{i_1}$, with w_1, w_2, w_3, w_4 real numbers; then $w = w_1 + w_2\mathbf{i_1} + w_3\mathbf{i_2} + w_4\mathbf{i_1}\mathbf{i_2}$.

Set

$$\mathbf{i_2}\mathbf{i_1} = \mathbf{i_1}\mathbf{i_2} =: \mathbf{j}.$$

This bicomplex number has the following properties which come directly from the definitions:

$$\mathbf{i_2}\mathbf{j} = \mathbf{j}\mathbf{i_2} = -\mathbf{i_1}, \quad \mathbf{i_1}\mathbf{j} = \mathbf{j}\mathbf{i_1} = -\mathbf{i_2}, \quad \mathbf{j}^2 = \mathbf{i_0} = 1.$$

Recalling that $\mathbf{i_1}$, $\mathbf{i_2}$ have the name of imaginary units, we shall attribute to $\mathbf{j}$ that of the *hyperbolic* (imaginary) unit. In particular, the set

$$\mathbb{D} := \{z_1 + z_2\mathbf{j} \mid z_1, z_2 \in \mathbb{R}\}$$

will be called the set of hyperbolic numbers (also called duplex numbers). With all this we see that the set $\mathbb{T}$ can be seen as

$$\mathbb{T} := \{w_1\mathbf{i_0} + w_2\mathbf{i_1} + w_3\mathbf{i_2} + w_4\mathbf{j} \mid w_1, w_2, w_3, w_4 \in \mathbb{R}\}.$$

If $w_3 = w_4 = 0$ then the bicomplex number $w = w_1\mathbf{i_0} + w_2\mathbf{i_1} + 0\mathbf{i_2} + 0\mathbf{j}$ is identified with the complex number $w_1 + w_2\mathbf{i_1} \in \mathbb{C}(\mathbf{i_1})$ and if $w_2 = w_4 = 0$ then the bicomplex number $w = w_1\mathbf{i_0} + 0\mathbf{i_1} + w_3\mathbf{i_2} + 0\mathbf{j}$ is identified with the complex number $w_1 + w_3\mathbf{i_2} \in \mathbb{C}(\mathbf{i_2})$, where saying "complex number" we are referring to what is explained in Sections 1.2 and 1.3. The bicomplex number $w = w_1\mathbf{i_0} + 0\mathbf{i_1} + 0\mathbf{i_2} + 0\mathbf{j}$ is identified with the real number w_1, thus, under such an identification, $\mathbb{R}$ is a subset of $\mathbb{T}$.

1.5

The Caley table of the set $\mathbb{T}$ is of the form:

$\cdot$	$\mathbf{i_0}$	$\mathbf{i_1}$	$\mathbf{i_2}$	$\mathbf{j}$
$\mathbf{i_0}$	$\mathbf{i_0}$	$\mathbf{i_1}$	$\mathbf{i_2}$	$\mathbf{j}$
$\mathbf{i_1}$	$\mathbf{i_1}$	-$\mathbf{i_0}$	$\mathbf{j}$	-$\mathbf{i_2}$
$\mathbf{i_2}$	$\mathbf{i_2}$	$\mathbf{j}$	-$\mathbf{i_0}$	-$\mathbf{i_1}$
$\mathbf{j}$	$\mathbf{j}$	-$\mathbf{i_2}$	-$\mathbf{i_1}$	$\mathbf{i_0}$

i.e., $\mathbf{i_0} := 1$ acts as identity, and

$$\begin{aligned} \mathbf{i_1}\mathbf{i_2} &= \mathbf{i_2}\mathbf{i_1} = \mathbf{j}, \\ \mathbf{i_1}\mathbf{j} &= \mathbf{j}\mathbf{i_1} = -\mathbf{i_2}, \\ \mathbf{i_2}\mathbf{j} &= \mathbf{j}\mathbf{i_2} = -\mathbf{i_1}, \\ \mathbf{j}^2 &= \mathbf{i_0}. \end{aligned}$$

Such an approach says that we endowed the real four-dimensional linear space $\mathbb{R}^4$ with a canonical basis

$$\mathbf{i_0} = (1,0,0,0),\ \mathbf{i_1} = (0,1,0,0),\ \mathbf{i_2} = (0,0,1,0),\ \mathbf{j} = (0,0,0,1),$$

with the following arithmetic operations: if $s = s_1 \cdot 1 + s_2\mathbf{i_1} + s_3\mathbf{i_2} + s_4\mathbf{j}$ and $t = t_1 \cdot 1 + t_2\mathbf{i_1} + t_3\mathbf{i_2} + t_4\mathbf{j}$ then

$$\begin{aligned} s+t &:= (s_1+t_1)\cdot 1 + (s_2+t_2)\mathbf{i_1} + (s_3+t_3)\mathbf{i_2} + (s_4+t_4)\mathbf{j}; \\ s\cdot t &:= \quad (s_1t_1 - s_2t_2 - s_3t_3 + s_4t_4) \\ &\quad +(s_1t_2 + s_2t_1 - s_3t_4 - s_4t_3)\mathbf{i_1} \\ &\quad +(s_1t_3 - s_2t_4 + s_3t_1 - s_4t_2)\mathbf{i_2} \\ &\quad +(s_1t_4 + s_2t_3 + s_3t_2 + s_4t_1)\mathbf{j}, \end{aligned}$$

which makes $\mathbb{T}$, as we already know, a commutative ring with unit and with zero divisors.

1.6

Introduce the following bicomplex numbers:

$$\mathbf{e_1} = \frac{1+\mathbf{i_1}\mathbf{i_2}}{2}; \qquad \mathbf{e_2} = \frac{1-\mathbf{i_1}\mathbf{i_2}}{2};$$

they have the easily verifiable properties:

$$\mathbf{e_1}^2 = \mathbf{e_1}; \quad \mathbf{e_2}^2 = \mathbf{e_2}; \quad \mathbf{e_1} + \mathbf{e_2} = 1; \quad \mathbf{e_1}\mathbf{e_2} = \mathbf{e_2}\mathbf{e_1} = 0;$$

which means that $\mathbf{e_1}$ and $\mathbf{e_2}$ are idempotents (sometimes called also orthogonal idempotents because of the last property). They allow us to obtain a decomposition of the set $\mathbb{T}$: for any $z_1 + z_2\mathbf{i_2} \in \mathbb{T}$

$$z_1 + z_2\mathbf{i_2} = (z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2}.$$

The properties of $\mathbf{e_1}$, $\mathbf{e_2}$ ensures that the decomposition is unique, hence the two mappings:

$$\begin{aligned} P_1 : z_1 + z_2\mathbf{i_2} \in \mathbb{T} \mapsto (z_1 - z_2\mathbf{i_1}) \in \mathbb{C}(\mathbf{i_1}), \\ P_2 : z_1 + z_2\mathbf{i_2} \in \mathbb{T} \mapsto (z_1 + z_2\mathbf{i_1}) \in \mathbb{C}(\mathbf{i_1}), \end{aligned}$$

define a pair of mutually complementary projections:

$$[P_1]^2 = P_1, \quad [P_2]^2 = P_2,$$

$$P_1\mathbf{e_1} + P_2\mathbf{e_2} = Id,$$

Id being the identity operator. They have several remarkable properties. Given $u, v \in \mathbb{T}$,

$$\begin{array}{rcl} P_1(u+v) & = & P_1(u) + P_1(v), \\ P_2(u+v) & = & P_2(u) + P_2(v), \\ P_1(u \cdot v) & = & P_1(u) \cdot P_1(v), \\ P_2(u \cdot v) & = & P_2(u) \cdot P_2(v). \end{array}$$

Those properties give an opportunity to reduce certain operations to the component-wise operations which we shall use many times in what follows.

2 The conjugations in bicomplex numbers

The complex conjugation plays an extremely important role for both algebraic and geometric properties of $\mathbb{C}$, and for analysis of complex functions. It appears that there are three conjugations in $\mathbb{T}$ which is not surprising: indeed, the complex conjugation is totally given by its action over the imaginary unit; thus one expects at least two conjugations on $\mathbb{T}$ but one more candidate could arise from composing them. Consider these ideas in more detail.

Note: The proofs of the properties presented in this section are rather technical and very simple, so we omitted them.

2.1 Definition (bicomplex conjugation with respect to $\mathbf{i_1}$).

It is determined by the formula

$$(z_1 + z_2\mathbf{i_2})^\star := \overline{z_1} + \overline{z_2}\mathbf{i_2}$$

for all $z_1, z_2 \in \{x + y\mathbf{i_1} \mid \mathbf{i_1}^2 = -1\}$ where $\overline{z_1}$, $\overline{z_2}$ are complex conjugate of complex numbers z_1, z_2.

We shall call the above conjugation, sometimes, 1st kind of conjugation of bicomplex numbers.

2.2 Proposition (properties of bicomplex conjugation with respect to $\mathbf{i_1}$).

a) $(s+t)^\star = s^\star + t^\star, \qquad \forall s, t \in \mathbb{T}.$

b) $(s-t)^\star = s^\star - t^\star, \qquad \forall s, t \in \mathbb{T}.$

c) $(w^\star)^\star = w, \qquad \forall w \in \mathbb{T}.$

d) $(s \cdot t)^\star = s^\star \cdot t^\star, \qquad \forall s, t \in \mathbb{T}.$

2.3 Definition (bicomplex conjugation with respect to $\mathbf{i_2}$; or 2nd kind of conjugation).

It is determined by the formula:

$$(z_1 + z_2\mathbf{i_2})^* := z_1 - z_2\mathbf{i_2}, \qquad \forall z_1, z_2 \in \mathbb{C}(\mathbf{i_1}).$$

2.4 Proposition (properties of the bicomplex conjugation with respect to $\mathbf{i_2}$).

a) $(s+t)^* = s^* + t^*, \qquad \forall s, t \in \mathbb{T}.$

b) $(s-t)^* = s^* - t^*, \qquad \forall s, t \in \mathbb{T}.$

c) $(w^*)^* = w, \qquad \forall w \in \mathbb{T}.$

d) $(s \cdot t)^* = s^* \cdot t^*, \qquad \forall s, t \in \mathbb{T}.$

2.5 Definition (3rd kind of conjugation).

It is a composition of the above two conjugations, and it is defined by the formula:

$$(z_1 + z_2\mathbf{i_2})^\dagger := (z_1 + z_2\mathbf{i_2}^\star)^* = (z_1 + z_2\mathbf{i_2}^*)^\star = \overline{z_1} - \overline{z_2}\mathbf{i_2}.$$

2.6 Proposition (properties of the 3rd kind of conjugation).

For all s, t, and $w \in \mathbb{T}$:

a) $(s+t)^\dagger = s^\dagger + t^\dagger, \quad \forall s, t \in \mathbb{T}.$

b) $(s-t)^\dagger = s^\dagger - t^\dagger, \quad \forall s, t \in \mathbb{T}.$

b) $(w^\dagger)^\dagger = w, \quad \forall w \in \mathbb{T}.$

d) $(s \cdot t)^\dagger = s^\dagger \cdot t^\dagger, \quad \forall s, t \in \mathbb{T}.$

2.6.1

On the subsets $\mathbb{C}(\mathbf{i_1})$ and $\mathbb{C}(\mathbf{i_2})$ of $\mathbb{T}$, the 3rd conjugation acts as the "complex conjugation" on the copies of the set $\mathbb{C}$; indeed let $w_1 = z_1 \in \mathbb{C}(\mathbf{i_1})$, then

$$\begin{aligned} w_1^\dagger &= (z_1 + 0\mathbf{i_2})^\dagger \\ &= \overline{z_1} - \overline{0}\mathbf{i_2} \\ &= \overline{z_1}; \end{aligned}$$

in a similar way let $w_2 = x + y\mathbf{i_2}$ with $x, y \in \mathbb{R}$, then

$$\begin{aligned} w_2^\dagger &= (x + y\mathbf{i_2})^\dagger \\ &= x - y\mathbf{i_2} \\ &= \overline{x + y\mathbf{i_2}}. \end{aligned}$$

Of course, we abuse a little bit here denoting with the same symbol " $^-$ " two different, strictly speaking, operations, one acts on $\mathbb{C}(\mathbf{i_1})$ and the other on $\mathbb{C}(\mathbf{i_2})$. We believe that this does not cause any confusion and we are going to keep denoting by $\overline{z}$ the conjugation in any of the set $\mathbb{C}(\mathbf{k}) := \{x + \mathbf{k}y \mid x, y \in \mathbb{R}\}$ with $\mathbf{k}^2 = \pm 1$: if $z \in \mathbb{C}(\mathbf{k})$ then $\overline{z} := x - \mathbf{k}y$. Note that $(x + y\mathbf{j})^\dagger = x + y\mathbf{j}$ $\forall x, y \in \mathbb{R}$.

2.7

All the above has a convenient expression in term of the following operations:

$$\begin{aligned} Z_1 :& \quad w \in \mathbb{T} \mapsto w^\star; \\ Z_2 :& \quad w \in \mathbb{T} \mapsto w^*; \end{aligned}$$

Indeed, Proposition 2.2 says that Z_1 is an additive and multiplicative involution on the ring $\mathbb{T}$, while Proposition 2.4 says that the same is true for Z_2.

2.8

It is worthwhile to note that although the sets $\mathbb{C}(\mathbf{i_1})$ and $\mathbb{C}(\mathbf{i_2})$ seem to be absolutely equivalent inside $\mathbb{T}$, the definitions of all the three conjugations reflect certain asymmetry between them. Indeed, Definition 2.1 takes $\mathbb{C}(\mathbf{i_1})$ as the subjacent set and extends its "complex conjugation" onto the set $\mathbb{C}^2$ seen as $\mathbb{C}(\mathbf{i_1}) \times \mathbb{C}(\mathbf{i_1})$, without affecting the additional structure generated by $\mathbf{i_2}$; this type of conjugation is the one considered in classical function theory in $\mathbb{C}^2$: the conjugation there is just the component-wise conjugation. But we shall see below that this conjugation does not correspond to some essential structures of $\mathbb{T}$. In contrast to the above said, the 2nd conjugation do is generated by $\mathbf{i_2}$, and the result could hardly be seen from inside the classical approach to $\mathbb{C}^2$: why the operation $(z_1, z_2) \in \mathbb{C}^2 \mapsto (z_1, -z_2)$ is better than many other possible? It's appeared that combining both conjugation make sense also which will be demonstrated below.

3 The different moduli in $\mathbb{T}$

For complex numbers the product of a complex number with its conjugate gives the square of the Euclidean metric in $\mathbb{R}^2$ which proved to be a very deep fact. Thus consider now the analogues of it for bicomplex numbers. We have for

$w = z_1 + z_2\mathbf{i_2}$:

$$\begin{aligned} w \cdot w^* &= z_1^2 + z_2^2, && (1) \\ w \cdot w^\star &= (|z_1|^2 - |z_2|^2) + 2\mathrm{Re}(z_1\overline{z_2})\mathbf{i_2}, && (2) \\ w \cdot w^\dagger &= (|z_1|^2 + |z_2|^2) - 2\mathrm{Im}(z_1\overline{z_2})\mathbf{j}. && (3) \end{aligned}$$

The equality (1) plays the role of the equality $z \cdot \overline{z} = x^2 + y^2$ for complex numbers with the complex-valued form $z_1^2 + z_2^2$ being a substitute for the Euclidean metric of the complex plane $\mathbb{C}$. Note also that the real part in (3) is the Euclidean metric in $\mathbb{C}^2 \cong \mathbb{T} = \mathbb{C}_2$, while the real part in (2) is related to the hyperbolic structure. Moreover, $w \cdot w^* \in \mathbb{C}(\mathbf{i_1})$, $w \cdot w^\star \in \mathbb{C}(\mathbf{i_2})$ and $w \cdot w^\dagger \in \mathbb{D}$. This brings us to the following definitions.

3.1 Definition

Given a bicomplex number $w = z_1 + z_2\mathbf{i_2}$ with $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$, then

a) the real number

$$|w| = \sqrt{|z_1|^2 + |z_2|^2}$$

will be referred to as the *real modulus* of $w = z_1 + z_2\mathbf{i_2}$;

b) the complex number (in $\mathbb{C}(\mathbf{i_1})$)

$$|w|_{\mathbf{i_1}} := \begin{cases} |w|_c = \sqrt{z_1^2 + z_2^2} & \text{if } w = z_1 + z_2\mathbf{i_2}, \\ |w|_h = \sqrt{\zeta_1^2 - \zeta_2^2} & \text{if } w = \zeta_1 + \zeta_2\mathbf{j}, \end{cases}$$

will be referred to as the $\mathbf{i_1}$-*modulus*. Depending on the way to express w, $|w|_{\mathbf{i_1}} = |w|_c$ (called *the complex* $\mathbf{i_1}$*-modulus*) or $|w|_{\mathbf{i_1}} = |w|_h$ (called *the hyperbolic modulus*).

c) The complex number (in $\mathbb{C}(\mathbf{i_2})$)

$$|w|_{\mathbf{i_2}} := \sqrt{(|z_1|^2 - |z_2|^2) + 2\mathrm{Re}(z_1\overline{z_2})\mathbf{i_2}},$$

will be referred to as the $\mathbf{i_2}$-*modulus* of $w = z_1 + z_2\mathbf{i_2}$. In particular,

$$|w|_{\mathbf{i_2}} = \sqrt{\gamma_1^2 + \gamma_2^2} \quad \text{if } w = \gamma_1 + \gamma_2\mathbf{i_1}$$

where $\gamma_1, \gamma_2 \in \mathbb{C}(\mathbf{i_2})$.

d) The hyperbolic number

$$|w|_{\mathbf{j}} := |z_1 - z_2\mathbf{i_1}|\mathbf{e_1} + |z_1 + z_2\mathbf{i_1}|\mathbf{e_2}$$

will be referred to as the $\mathbf{j}$-*modulus* of $w = z_1 + z_2\mathbf{i_2}$.

In the above formulas, the square root of complex numbers (in $\mathbb{C}(\mathbf{i_1})$ or $\mathbb{C}(\mathbf{i_2})$) is understood as one of two possible values which is taken with non-negative real part: given $\alpha, \beta \in \mathbb{R}$, of two values

$$(\alpha + \beta\mathbf{i})^{\frac{1}{2}} = \pm \left(\sqrt{\frac{\alpha + \sqrt{\alpha^2+\beta^2}}{2}} + \mathbf{i} \cdot \operatorname{sgn}(\beta) \sqrt{\frac{-\alpha + \sqrt{\alpha^2+\beta^2}}{2}} \right)$$

we choose

$$\sqrt{\alpha + \beta\mathbf{i}} := \sqrt{\frac{\alpha + \sqrt{\alpha^2+\beta^2}}{2}} + \mathbf{i} \cdot \operatorname{sgn}(\beta) \sqrt{\frac{-\alpha + \sqrt{\alpha^2+\beta^2}}{2}}$$

where

$$\operatorname{sgn}(\beta) := \begin{cases} 1 & \text{if } \beta \geq 0, \\ -1 & \text{if } \beta < 0. \end{cases}$$

Furthermore, the **j**-modulus can be well justified as follows: first,

$$\begin{aligned} (z_1 + z_2\mathbf{i_2})^\dagger &= \overline{z_1} - \overline{z_2}\mathbf{i_2} \\ &= (\overline{z_1} + \overline{z_2}\mathbf{i_1})\mathbf{e_1} + (\overline{z_1} - \overline{z_2}\mathbf{i_1})\mathbf{e_2} \\ &= \overline{(z_1 - z_2\mathbf{i_1})}\mathbf{e_1} + \overline{(z_1 + z_2\mathbf{i_1})}\mathbf{e_2}, \end{aligned}$$

hence,

$$w \cdot w^\dagger = |z_1 - z_2\mathbf{i_1}|^2\mathbf{e_1} + |z_1 + z_2\mathbf{i_1}|^2\mathbf{e_2} \in \mathbb{D}$$

and

$$\sqrt{w \cdot w^\dagger} = \sqrt{|z_1 - z_2\mathbf{i_1}|^2\mathbf{e_1} + |z_1 + z_2\mathbf{i_1}|^2\mathbf{e_2}},$$

where our choice of the square root is:

$$\begin{aligned} \sqrt{|z_1 - z_2\mathbf{i_1}|^2\mathbf{e_1} + |z_1 + z_2\mathbf{i_1}|^2\mathbf{e_2}} &:= \sqrt{|z_1 - z_2\mathbf{i_1}|^2}\mathbf{e_1} + \sqrt{|z_1 + z_2\mathbf{i_1}|^2}\mathbf{e_2} \\ &= |z_1 - z_2\mathbf{i_1}|\mathbf{e_1} + |z_1 + z_2\mathbf{i_1}|\mathbf{e_2}. \end{aligned}$$

Thus, for $w = z_1 + z_2\mathbf{i_2}$, we may conclude that:

$$\begin{aligned} |w|_{\mathbf{i_1}} = |w|_c &= \sqrt{w \cdot w^*}, \\ |w|_{\mathbf{i_2}} &= \sqrt{w \cdot w^\star}, \\ |w|_{\mathbf{j}} &= \sqrt{w \cdot w^\dagger} \end{aligned}$$

and

$$|w| = \sqrt{\operatorname{Re}(w \cdot w^\dagger)} = \sqrt{\operatorname{Re}(|w|_{\mathbf{j}}^2)}.$$

3.2

It appears that for a bicomplex number to be invertible or not is related to its complex, not real, modulus. Indeed, consider (1). If $|w|_c \neq 0$ then

$$w \cdot \frac{w^*}{|w|_c^2} = \frac{w^*}{|w|_c^2} \cdot w = 1$$

which means that for such w there exists its inverse w^{-1} and it is given by

$$w^{-1} = \frac{w^*}{|w|_c^2}, \tag{4}$$

compare with the formula of the inverse for complex numbers.

Assume now that $w \neq 0$ but $|w|_c = 0$; then $w^* \neq 0$ but (1) says that

$$w \cdot w^* = 0,$$

which means that w cannot be invertible since being invertible would imply that $w^* = 0$. Denote by $\mathbb{T}^{-1}$ the set of all invertible elements in $\mathbb{T}$, we have just proved the following

3.2.1 Theorem (invertible bicomplex numbers).

$$\mathbb{T}^{-1} = \{w = z_1 + z_2\mathbf{i_2} \mid z_1^2 + z_2^2 = |w|_c^2 \neq 0\}.$$

Some immediate consequences of this are:

3.2.2 Corollary

a) Let s and $t \in \mathbb{T}$, if s and t are invertible, then st is also invertible and $(st)^{-1} = s^{-1}t^{-1}$.

b) Let $w \in \mathbb{T}$, w is invertible if and only if w^* is also invertible; besides $(w^*)^{-1} = (w^{-1})^*$.

3.2.3

The set $\mathcal{NC} = \{z_1 + z_2\mathbf{i_2} \mid z_1^2 + z_2^2 = 0\} = \{z_1 + z_2\mathbf{j} \mid z_1^2 - z_2^2 = 0\}$ is called the set of *zero divisors* of $\mathbb{T}$, or equivalently, the *null-cone*. We can write also:

$$\mathcal{NC} = \mathcal{O}_2 = \{z(\mathbf{i_1} \pm \mathbf{i_2}) \mid z \in \mathbb{C}(\mathbf{i_1})\}$$

where $\mathbf{i_1} \pm \mathbf{i_2}$ are the "basic" zero divisors, $(\mathbf{i_1} + \mathbf{i_2})(\mathbf{i_1} - \mathbf{i_2}) = 0$. In fact, $w \in \mathbb{T}$ belongs to the null-cone if and only if at least one of $P_1(w)$ and $P_2(w)$ is equal to zero. Moreover, if $u \in \mathbb{T}^{-1}$ then

$$P_1(u^{-1}) = (P_1(u))^{-1}, \qquad P_2(u^{-1}) = (P_2(u))^{-1}.$$

3.2.4 Corollary

Let $w \in \mathbb{T} \setminus \{0\}$ and $w \in \mathbb{C}(\mathbf{i_1})$ or $w \in \mathbb{C}(\mathbf{i_2})$, then $w \in \mathbb{T}^{-1}$.

In other words, $(\mathbb{C}(\mathbf{i_1}) \cup \mathbb{C}(\mathbf{i_2})) \setminus \{0\} \subset \mathbb{T}^{-1}$. What is happening in $\mathbb{D}$ will be explained later.

4 Properties of moduli

4.1 Real modulus

It is already well known that the function $| \ | : \mathbb{T} \to \mathbb{R}$ is a norm on the real space $\mathbb{R}^4 \cong \mathbb{T}$, i.e. $\forall s, t \in \mathbb{T}$ and $a \in \mathbb{R}$:

(1) $|s| \geq 0$,
(2) $|s| = 0 \Leftrightarrow s = 0$,
(3) $|a \cdot s| = |a||s|$,
(4) $|s + t| \leq |s| + |t|$.

Since, the space $\mathbb{R}^4$ with the Euclidean norm is known to be a complete space, then the normed $\mathbb{R}$-linear space $\mathbb{T}$ is a complete space. More precisely, $(\mathbb{T}, +, \cdot, | \ |)$ is a real Banach space.

4.1.1 Lemma

Let $s = s_1 + s_2\mathbf{i_1} + s_3\mathbf{i_2} + s_4\mathbf{j} \in \mathbb{T}$ and $t = t_1 + t_2\mathbf{i_1} + t_3\mathbf{i_2} + t_4\mathbf{j} \in \mathbb{T}$. Then,

$$|s \cdot t|^2 - |s|^2|t|^2 = 4(s_1s_4 - s_2s_3)(t_1t_4 - t_2t_3), \tag{5}$$

or equivalently,

$$|st|^2 = |s|^2 \cdot |t|^2 + 4 \begin{vmatrix} s_1 & s_2 \\ s_3 & s_4 \end{vmatrix} \begin{vmatrix} t_1 & t_2 \\ t_3 & t_4 \end{vmatrix}.$$

Proof. We have that

$$\begin{aligned}
|s \cdot t|^2 - |s|^2|t|^2 &= (s_1t_1 + s_4t_4 - s_2t_2 - s_3t_3)^2 + (s_1t_2 + s_2t_1 - s_3t_4 - s_4t_3)^2 \\
&+ (s_1t_3 + s_3t_1 - s_2t_4 - s_4t_2)^2 + (s_1t_4 + s_2t_3 + s_3t_2 + s_4t_1)^2 \\
&- ((s_1^2 + s_2^2 + s_3^2 + s_4^2)(t_1^2 + t_2^2 + t_3^2 + t_4^2)) \\
&= 4s_1s_4t_1t_4 - 4t_1t_4s_2s_3 + 4s_2s_3t_2t_3 - 4s_1s_4t_2t_3 \\
&= 4t_1t_2(s_1s_4 - s_2s_3) + 4t_2t_3(s_2s_3 - s_1s_4) \\
&= 4(s_1s_4 - s_2s_3)(t_1t_4 - t_2t_3).\square
\end{aligned}$$

4.1.2 Theorem

Let $s = s_1 + s_2\mathbf{i_1} + s_3\mathbf{i_2} + s_4\mathbf{j} \in \mathbb{T}$ and $t = t_1 + t_2\mathbf{i_1} + t_3\mathbf{i_2} + t_4\mathbf{j} \in \mathbb{T}$. Then:

$$|s \cdot t| = |s||t| \text{ if and only if one of the matrices } \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \text{ and } \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}$$

is degenerated.

$$|s \cdot t| > |s||t| \text{ if and only if } \operatorname{sgn}\begin{vmatrix} s_1 & s_2 \\ s_3 & s_4 \end{vmatrix} = \operatorname{sgn}\begin{vmatrix} t_1 & t_2 \\ t_3 & t_4 \end{vmatrix}.$$

$$|s \cdot t| < |s||t| \text{ if and only if } \operatorname{sgn}\begin{vmatrix} s_1 & s_2 \\ s_3 & s_4 \end{vmatrix} = -\operatorname{sgn}\begin{vmatrix} t_1 & t_2 \\ t_3 & t_4 \end{vmatrix}.$$

Proof. The proof is a direct consequence of the Lemma 4.1.1. □

In particular, there holds:

4.1.3 Corollary

Let $w \in \mathbb{T}$ and $z \in \mathbb{C}(\mathbf{i_1})$ or $\mathbb{C}(\mathbf{i_2})$. Then $|z \cdot w| = |z||w|$.

The following theorem is quite important.

4.1.4 Theorem

Let $s, t \in \mathbb{T}$ then $|s \cdot t| \leq \sqrt{2}|s||t|$.

Proof. Let $s = z_1 + z_2\mathbf{i_2} \in \mathbb{T}$ and $t = z_3 + z_4\mathbf{i_2} \in \mathbb{T}$, then

$$s \cdot t = (z_1 + z_2\mathbf{i_2}) \cdot (z_3 + z_4\mathbf{i_2}) = z_1(z_3 + z_4\mathbf{i_2}) + z_2(z_3 + z_4\mathbf{i_2})\mathbf{i_2}.$$

Moreover, $|z_1(z_3 + z_4\mathbf{i_2})| = |z_1||z_3 + z_4\mathbf{i_2}|$ and $|z_2(z_3 + z_4\mathbf{i_2})\mathbf{i_2}| = |z_2||(z_3 + z_4\mathbf{i_2})\mathbf{i_2}| = |z_2||(z_3 + z_4\mathbf{i_2})||\mathbf{i_2}| = |z_2||(z_3 + z_4\mathbf{i_2})|$. Therefore, from the triangle inequality, we have that

$$\begin{aligned} |s \cdot t| = |(z_1 + z_2\mathbf{i_2}) \cdot (z_3 + z_4\mathbf{i_2})| &\leq |z_1||z_3 + z_4\mathbf{i_2}| + |z_2||(z_3 + z_4\mathbf{i_2})| \\ &\leq (|z_1| + |z_2|)|z_3 + z_4\mathbf{i_2}|. \end{aligned}$$

Since $2|z_1||z_2| \leq |z_1|^2 + |z_2|^2$, then $(|z_1| + |z_2|)^2 \leq 2(|z_1|^2 + |z_2|^2)$. Hence,

$$(|z_1| + |z_2|) \leq \sqrt{2}(|z_1|^2 + |z_2|^2)^{1/2}.\Box$$

Now, from the following observation:

$$|e_i \cdot e_i| = |e_i| = \frac{\sqrt{2}}{2} = \sqrt{2}|e_i||e_i|, \quad i = 1, 2,$$

we note that the constant $\sqrt{2}$ is the best possible one in Theorem 4.1.4. Moreover, if we combine the last results with the fact that $(\mathbb{T}, +, \cdot, |\ |)$ is a Banach space, we obtain that $(\mathbb{T}, +, \cdot, |\ |)$ is a modified complex Banach algebra.

Finally, we conclude this subsection with the following analog of the Pythagoras Theorem:

4.1.5 Theorem

Let $z_1 + z_2\mathbf{i_2} \in \mathbb{T}$, then

$$|z_1 + z_2\mathbf{i_2}| = \left(\frac{|z_1 - z_2\mathbf{i_1}|^2 + |z_1 + z_2\mathbf{i_1}|^2}{2}\right)^{1/2},$$

or equivalently, but in a more impressive form,

$$2|w|^2 = |P_1(w)|^2 + |P_2(w)|^2.$$

Proof. Let $z_1 = x_1 + y_1\mathbf{i_1}$ and $z_2 = x_2 + y_2\mathbf{i_1}$. Then $z_1 - z_2\mathbf{i_1} = (x_1 + y_2) + (y_1 - x_2)\mathbf{i_1}$ and $z_1 + z_2\mathbf{i_1} = (x_1 - y_2) + (y_1 + x_2)\mathbf{i_1}$. Therefore,

$$\begin{aligned} |z_1 - z_2\mathbf{i_1}|^2 + |z_1 + z_2\mathbf{i_1}|^2 &= (x_1 + y_2)^2 + (y_1 - x_2)^2 \\ &+ (x_1 - y_2)^2 + (y_1 + x_2)^2 \\ &= 2(x_1^2 + y_1^2 + x_2^2 + y_2^2) \\ &= 2(|z_1|^2 + |z_2|^2) \\ &= 2|z_1 + z_2\mathbf{i_2}|^2. \Box \end{aligned}$$

4.2 $\mathbf{i_1}$-modulus : *Complex and Hyperbolic versions*

4.2.1 Proposition

Let $w \in \mathbb{T}$. Then $|w|_c = 0$ if and only if $w \in \mathcal{O}_2$.

Proof. It is enough to compare both definitions, those of complex modulus and of $\mathcal{O}_2$, see Section 3.2.3.

$\Box$

4.2.2 Corollary

Let $w \in \mathbb{T}$. Then $||w|_c| = 0$ if and only if $w \in \mathcal{O}_2$.

Proof. We know that $|w|_c = 0 \Leftrightarrow |w|_c^2 = z_1^2 + z_2^2 = 0$. Therefore, $||w|_c| = 0 \Leftrightarrow ||w|_c^2| = |z_1^2 + z_2^2| = 0$. Moreover, $|z_1^2 + z_2^2| = 0 \Leftrightarrow z_1 + z_2\mathbf{i_2} \in \mathcal{O}_2$. Hence, $||w|_c| = 0$ if and only if $w \in \mathcal{O}_2$.$\Box$

4.2.3 Proposition

Let $s \in \mathbb{T}$ and $t \in \mathbb{T}$. Then $|s \cdot t|_c^2 = |s|_c^2 \cdot |t|_c^2$.

Proof. Let $s = z_1 + z_2\mathbf{i_2}$ and $t = z_3 + z_4\mathbf{i_2}$. Hence,

$$\begin{array}{rcl} |s \cdot t|_c^2 & = & |(z_1 + z_2\mathbf{i_2}) \cdot (z_3 + z_4\mathbf{i_2})|_c^2 \\ & = & |(z_1z_3 - z_2z_4) + (z_1z_4 + z_2z_3)\mathbf{i_2}|_c^2 \\ & = & (z_1z_3 - z_2z_4)^2 + (z_1z_4 + z_2z_3)^2 \\ & = & (z_1z_3)^2 + (z_2z_4)^2 + (z_1z_4)^2 + (z_2z_3)^2 \\ & = & |(z_1 + z_2\mathbf{i_2})|_c^2 \cdot |(z_3 + z_4\mathbf{i_2})|_c^2 \\ & = & |s|_c^2 \cdot |t|_c^2 . \square \end{array}$$

4.2.4 Corollary

Let $s \in \mathbb{T}$ and $t \in \mathbb{T}$. Then $||s \cdot t|_c| = ||s|_c| \cdot ||t|_c|$.

Proof. From Proposition 4.2.3 we have that $||s \cdot t|_c^2| = ||s|_c^2 \cdot |t|_c^2| = ||s|_c^2| \cdot ||t|_c^2|$. Therefore,

$$\begin{array}{rcl} ||s \cdot t|_c|^2 & = & ||s|_c|^2 \cdot ||t|_c|^2 \\ & = & (||s|_c| \cdot ||t|_c|)^2, \end{array}$$

and this is all. $\square$

4.2.5 Proposition

Let $w \in \mathbb{T}^{-1}$. Then $||w|_c|^{-1} = ||w^{-1}|_c|$.

Proof. Let $w = z_1 + z_2\mathbf{i_2}$ be invertible, hence $||w|_c^{-1}| = \left|\frac{1}{\sqrt{z_1^2+z_2^2}}\right|$. Moreover, $w^{-1} = \frac{z_1}{[z_1^2+z_2^2]} - \mathbf{i_2}\frac{z_2}{[z_1^2+z_2^2]}$. Thus,

$$\begin{array}{rcl} ||w^{-1}|_c| & = & \left|\sqrt{\frac{z_1^2}{[z_1^2+z_2^2]^2} + \frac{z_2^2}{[z_1^2+z_2^2]^2}}\right| \\ & = & \left|\sqrt{\frac{1}{z_1^2+z_2^2}}\right| \\ & = & \left|\frac{1}{\sqrt{z_1^2+z_2^2}}\right| \\ & = & ||w|_c^{-1}| . \square \end{array}$$

We can explicitly express a formula for $||w|_c|$.

4.2.6 Proposition

Let $w = w_1 + w_2\mathbf{i_1} + w_3\mathbf{i_2} + w_4\mathbf{j} \in \mathbb{T}$. Then

$$||w|_c| = \sqrt[4]{(w_1^2 - w_2^2 + w_3^2 - w_4^2)^2 + 4(w_1w_2 + w_3w_4)^2}.$$

Proof. Let $w = w_1 + w_2\mathbf{i_1} + w_3\mathbf{i_2} + w_4\mathbf{j} = z_1 + z_2\mathbf{i_2} \in \mathbb{T}$. Then,

$$\begin{aligned} |w|_c = \sqrt{z_1^2 + z_2^2} &:= \sqrt{\alpha + \beta\mathbf{i_1}} \\ &= \sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} + \mathbf{i_1} \cdot \operatorname{sgn}(\beta)\sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}}. \end{aligned}$$

Therefore, $||w|_c| = \sqrt{\sqrt{\alpha^2 + \beta^2}} = \sqrt[4]{\alpha^2 + \beta^2}$. We obtain the final result with this following fact:

$$z_1^2 + z_2^2 = (w_1^2 - w_2^2 + w_3^2 - w_4^2) + 2(w_1w_2 + w_3w_4)\mathbf{i_1} = \alpha + \beta\mathbf{i_1}.\Box$$

4.2.7

It is worth mentioning that the mapping $||\cdot|_c|$ does not meet the fundamental property:

$$||s + t|_c| \leq ||s|_c| + ||t|_c|.$$

For realizing this, it suffices to take $s := z(\mathbf{i_1} + \mathbf{i_2})$ and $t := z(\mathbf{i_1} - \mathbf{i_2})$ with $z \in \mathbb{C}(\mathbf{i_1}) \setminus \{0\}$ for which both s and t are in $\mathcal{O}_2$ but $s + t = 2z\mathbf{i_1}$ is invertible. In this case $|s|_c = |s|_c = 0$ but $|s + t|_c \neq 0$.

4.2.8

The hyperbolic modulus of a bicomplex number w is defined by its representation $w = \zeta_1 + \zeta_2\mathbf{j}$, see Definition 3.1, with ζ_1, ζ_2 in $\mathbb{C}(\mathbf{i_1})$. Assume that w has also the representation $w = z_1 + z_2\mathbf{i_2}$; it was shown above that $z_1 = \zeta_1$ and $z_2 = \mathbf{i_1}\zeta_2$, hence $|w|_h = |w|_c \in \mathbb{C}(\mathbf{i_1})$. That is why, the properties of the hyperbolic modulus are just reformulations of their analogs for the complex modulus, and we just enlist them below:

- Let $w \in \mathbb{T}$. Then $|w|_h = 0$ if and only if $w \in \mathcal{O}_2$.
- Let $w \in \mathbb{T}$. Then $||w|_h| = 0$ if and only if $w \in \mathcal{O}_2$.
- Let $s \in \mathbb{T}$ and $t \in \mathbb{T}$. Then $|s \cdot t|_h^2 = |s|_h^2 \cdot |t|_h^2$.
- Let $s \in \mathbb{T}$ and $t \in \mathbb{T}$. Then $||s \cdot t||_h = ||s|_h| \cdot ||t|_h|$.
- Let $w \in \mathbb{T}^{-1}$. Then $||w|_h|^{-1} = ||w^{-1}|_h|$.
- $w = w_1 + w_2\mathbf{i_1} + w_3\mathbf{i_2} + w_4\mathbf{j} \in \mathbb{T}$. Then

$$||w|_h| = \sqrt[4]{(w_1^2 + w_2^2 + w_3^2 + w_4^2)^2 + 4(w_1w_2 - w_3w_4)^2}.$$

4.3 $\mathbf{i_2}$-modulus

4.3.1 Proposition

Let $w \in \mathbb{T}$. Then $|w|_{\mathbf{i_2}} = 0$ if and only if $w \in \mathcal{O}_2$.

Proof. Let $w = z_1 + z_2\mathbf{i_2}$. From the definition of the modulus in $\mathbf{i_2}$: $|w|_{\mathbf{i_2}} \in \mathbb{C}(\mathbf{i_2})$. Therefore, $|w|_{\mathbf{i_2}} = 0 \Leftrightarrow |w|^2_{\mathbf{i_2}} = w \cdot w^\star = 0$. Hence, $|w|_{\mathbf{i_2}} = 0$ if and only if:

$$
\begin{aligned}
(z_1 + z_2\mathbf{i_2})(z_1 + z_2\mathbf{i_2})^\star &= (z_1 + z_2\mathbf{i_2})(\overline{z_1} + \overline{z_2}\mathbf{i_2}) \\
&= [(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2}] \\
&\quad \cdot[(\overline{z_1} - \overline{z_2}\mathbf{i_1})\mathbf{e_1} + (\overline{z_1} + \overline{z_2}\mathbf{i_1})\mathbf{e_2}] \\
&= (z_1 - z_2\mathbf{i_1})(\overline{z_1} - \overline{z_2}\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})(\overline{z_1} + \overline{z_2}\mathbf{i_1})\mathbf{e_2} \\
&= 0.
\end{aligned}
$$

Thus, $|w|_{\mathbf{i_2}} = 0$ if and only if $(z_1 - z_2\mathbf{i_1})(\overline{z_1} - \overline{z_2}\mathbf{i_1}) = 0$ and $(z_1 + z_2\mathbf{i_1})(\overline{z_1} + \overline{z_2}\mathbf{i_1}) = 0$, i.e. $w \in \mathcal{NC} = \{z_1 + z_2\mathbf{i_2} \mid z_1^2 + z_2^2 = 0\}$. $\square$

4.3.2 Proposition

Let $s \in \mathbb{T}$ and $t \in \mathbb{T}$. Then $|s \cdot t|^2_{\mathbf{i_2}} = |s|^2_{\mathbf{i_2}} \cdot |t|^2_{\mathbf{i_2}}$.

Proof. Let $s = z_1 + z_2\mathbf{i_2}$ and $t = z_3 + z_4\mathbf{i_2}$. Hence,

$$
\begin{aligned}
|s \cdot t|^2_{\mathbf{i_2}} &= |(z_1 + z_2\mathbf{i_2}) \cdot (z_3 + z_4\mathbf{i_2})|^2_{\mathbf{i_2}} \\
&= |(z_1z_3 - z_2z_4) + (z_1z_4 + z_2z_3)\mathbf{i_2}|^2_{\mathbf{i_2}} \\
&= [(z_1z_3 - z_2z_4) + (z_1z_4 + z_2z_3)\mathbf{i_2}][\overline{(z_1z_3 - z_2z_4)} + \overline{(z_1z_4 + z_2z_3)}\mathbf{i_2}] \\
&= (z_1z_3 - z_2z_4)(\overline{z_1z_3} - \overline{z_2z_4}) + (z_1z_3 - z_2z_4)(\overline{z_1z_4} + \overline{z_2z_3})\mathbf{i_2} \\
&= +(z_1z_4 + z_2z_3)(\overline{z_1z_3} - \overline{z_2z_4})\mathbf{i_2} - (z_1z_4 + z_2z_3)(\overline{z_1z_4} + \overline{z_2z_3}) \\
&= [z_1\overline{z_1} + z_1\overline{z_2}\mathbf{i_2} + \overline{z_1}z_2\mathbf{i_2} - z_2\overline{z_2}][z_3\overline{z_3} + z_3\overline{z_4}\mathbf{i_2} + \overline{z_3}z_4\mathbf{i_2} - z_4\overline{z_4}] \\
&= [(z_1 + z_2\mathbf{i_2})(\overline{z_1} + \overline{z_2}\mathbf{i_2})][(z_3 + z_4\mathbf{i_2})(\overline{z_3} + \overline{z_4}\mathbf{i_2})] \\
&= |(z_1 + z_2\mathbf{i_2})|^2_{\mathbf{i_2}} \cdot |(z_3 + z_4\mathbf{i_2})|^2_{\mathbf{i_2}} \\
&= |s|^2_{\mathbf{i_2}} \cdot |t|^2_{\mathbf{i_2}}.\square
\end{aligned}
$$

Finally, we obtain this following connection between the $\mathbf{i_1}$-*modulus* and the $\mathbf{i_2}$-*modulus*.

4.3.3 Proposition

Let $w \in \mathbb{T}$, then $||w|_{\mathbf{i_1}}| = ||w|_{\mathbf{i_2}}|$.

Proof. The proof can be established by a direct calculation using the definitions of the section 3.1. $\square$

4.4 j-modulus

4.4.1 Proposition

Let $w_1 \in \mathbb{C}(\mathbf{i_1})$ and $w_2 \in \mathbb{C}(\mathbf{i_2})$. Then

$$|w_1|_{\mathbf{j}} = |w_1| \text{ and } |w_2|_{\mathbf{j}} = |w_2|.$$

Proof. It follows directly from definitions and from Subsection 2.6.1.

4.4.2 Proposition

Let $w \in \mathbb{T}$. Then $|w|_{\mathbf{j}} = 0$ if and only if $w = 0$.
Proof. Let $w = z_1 + z_2\mathbf{i_2}$, then $|w|_{\mathbf{j}} \in \mathbb{C}(\mathbf{j})$. Therefore, $|w|_{\mathbf{j}} = 0$ if and only if:

$$\begin{aligned} \sqrt{(z_1 + z_2\mathbf{i_2})(z_1 + z_2\mathbf{i_2})^\dagger} &= |z_1 - z_2\mathbf{i_1}|\mathbf{e_1} + |z_1 + z_2\mathbf{i_1}|\mathbf{e_2} \\ &= 0. \end{aligned}$$

Thus, $|w|_{\mathbf{j}} = 0$ if and only if $|z_1 - z_2\mathbf{i_1}| = 0$ and $|z_1 + z_2\mathbf{i_1}| = 0$, i.e. $z_1 - z_2\mathbf{i_1} = 0$ and $z_1 + z_2\mathbf{i_1} = 0$□.

The next theorem shows that the real modulus of the **j**-modulus is the Euclidean distance in $\mathbb{C}^2 \cong \mathbb{T} = \mathbb{C}_2$.

4.4.3 Theorem

Let $w \in \mathbb{T}$. Then

$$||w|_{\mathbf{j}}| = |w| = \sqrt{\mathrm{Re}(|w|_{\mathbf{j}}^2)}.$$

Proof. From Theorem 4.1.5,

$$\begin{aligned} ||w|_{\mathbf{j}}| &= |(|z_1 - z_2\mathbf{i_1}|\mathbf{e_1} + |z_1 + z_2\mathbf{i_1}|\mathbf{e_2})| \\ &= \left(\frac{|(|z_1 - z_2\mathbf{i_1}|)|^2 + |(|z_1 + z_2\mathbf{i_1}|)|^2}{2}\right)^{1/2} \\ &= \left(\frac{|z_1 - z_2\mathbf{i_1}|^2 + |z_1 + z_2\mathbf{i_1}|^2}{2}\right)^{1/2} \\ &= |w|. \end{aligned}$$

Moreover, we already know that $|w| = \sqrt{\mathrm{Re}(w \cdot w^\dagger)} = \sqrt{\mathrm{Re}(|w|_{\mathbf{j}}^2)}$. Hence, $||w|_{\mathbf{j}}| = |w| = \sqrt{\mathrm{Re}(|w|_{\mathbf{j}}^2)}$.□

4.4.4 Proposition

Let $s \in \mathbb{T}$ and $t \in \mathbb{T}$. Then $|s \cdot t|_{\mathbf{j}} = |s|_{\mathbf{j}} |t|_{\mathbf{j}}$.

Proof. Let $s = z_1 + z_2\mathbf{i_2} \in \mathbb{T}$ and $t = z_3 + z_4\mathbf{i_2} \in \mathbb{T}$. Then

$$\begin{aligned}
|s \cdot t|_{\mathbf{j}} &= |[(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2}] \cdot [(z_3 - z_4\mathbf{i_1})\mathbf{e_1} + (z_3 + z_4\mathbf{i_1})\mathbf{e_2}]| \\
&= |(z_1 - z_2\mathbf{i_1})(z_3 - z_4\mathbf{i_1})|\mathbf{e_1} + |(z_1 + z_2\mathbf{i_1})(z_3 + z_4\mathbf{i_1})|\mathbf{e_2} \\
&= |(z_1 - z_2\mathbf{i_1})| \cdot |(z_3 - z_4\mathbf{i_1})|\mathbf{e_1} + |(z_1 + z_2\mathbf{i_1})| \cdot |(z_3 + z_4\mathbf{i_1})|\mathbf{e_2} \\
&= [|z_1 - z_2\mathbf{i_1}|\mathbf{e_1} + |z_1 + z_2\mathbf{i_1}|\mathbf{e_2}] \cdot [|z_3 - z_4\mathbf{i_1}|\mathbf{e_1} + |z_3 + z_4\mathbf{i_1}|\mathbf{e_2}] \\
&= |s|_{\mathbf{j}} |t|_{\mathbf{j}}.
\end{aligned}$$

□

5 Hyperbolic numbers

The ring $\mathbb{T}$ contains three two-dimensional subrings forming also real subalgebras: $\mathbb{C}(\mathbf{i_1})$, $\mathbb{C}(\mathbf{i_2})$ and $\mathbb{D}$. But each one of the first two is isomorphic to $\mathbb{C}$, and it does not make sense to study them separately, not in the context of the set of bicomplex numbers. This is not the case of $\mathbb{D}$ which is interesting and important by itself, and although formally the properties of hyperbolic numbers are "only" a particular case of their bicomplex antecedents we find it to be worthwhile to single them out explicitly. This will be done here for what we have obtained already, and we will keep doing this for the new facts.

5.1

Recall that $\mathbb{D} := \{x + y\mathbf{j} \mid \{x, y\} \subset \mathbb{R}, \mathbf{j}^2 = 1\}$. Of course, given two hyperbolic numbers $w_1 := x_1 + y_1\mathbf{j}$ and $w_2 := x_2 + y_2\mathbf{j}$ their sum and product are:

$$w_1 + w_2 := (x_1 + x_2) + (y_1 + y_2)\mathbf{j};$$

$$w_1 \cdot w_2 := (x_1x_2 + y_1y_2) + (x_1y_2 + x_2y_1)\mathbf{j}.$$

The Caley table has the form:

	1	**j**
1	1	**j**
j	**j**	1

The projections P_1 and P_2 introduced in Section 1.6 act on hyperbolic numbers as follows:

$$P_1 : x + y\mathbf{j} \in \mathbb{D} \mapsto (x + y) \in \mathbb{R},$$
$$P_2 : x + y\mathbf{j} \in \mathbb{D} \mapsto (x - y) \in \mathbb{R},$$

this is, there holds the following decomposition of hyperbolic numbers:

$$x + y\mathbf{j} = (x + y)\mathbf{e_1} + (x - y)\mathbf{e_2}.$$

5.2

Consider here what do the three bicomplex conjugations give for the specific case of hyperbolic numbers. We have:

$$(x + y\mathbf{j})^\star = \overline{z_1} + \overline{z_2} = x - \mathbf{i_1} y \mathbf{i_2} = x - y\mathbf{j}$$

and

$$(x + y\mathbf{j})^* = x - (y\mathbf{i_1})\mathbf{i_2} = x - y\mathbf{j},$$

i.e. the restrictions of both conjugations onto $\mathbb{D}$ coincide turning out to be a "natural conjugation" of hyperbolic numbers: if one defines Z_3 to be

$$Z_3 : w = x + y\mathbf{j} \in \mathbb{D} \mapsto x - y\mathbf{j}$$

then

$$Z_1|_{\mathbb{D}} = Z_2|_{\mathbb{D}} = Z_3.$$

The third conjugation being a composition of Z_1 and Z_2 becomes the identity operator on $\mathbb{D}$:

$$(x + y\mathbf{j})^\dagger = \left((x + y\mathbf{j})^\star\right)^* = (x - y\mathbf{j})^* = x + y\mathbf{j}.$$

5.2.1

Thus we have a unique conjugation on $\mathbb{D}$ which we shall denote sometimes as:

$$\overline{x + y\mathbf{j}} := x - y\mathbf{j}$$

and which acts involutively:

$$\overline{\overline{s}} = s,$$

additively:

$$\overline{s + t} = \overline{s} + \overline{t},$$

and multiplicatively:

$$\overline{s \cdot t} = \overline{s} \cdot \overline{t}.$$

5.3

The number of the moduli in this particular case reduces also. First of all, we have: for $w \in \mathbb{D}$

$$|w| = \sqrt{x^2 + y^2}$$

reflecting the Euclidean structure of $\mathbb{R}^2$. Then, for its $\mathbf{i_1}$-modulus one has

$$|w|_{\mathbf{i_1}} = \sqrt{x^2 - y^2}$$

and the same for the $\mathbf{i_2}$-modulus:

$$\begin{aligned} |w|_{\mathbf{i_2}} &= \sqrt{|z_1|^2 - |z_2|^2 + 2\mathrm{Re}(z_1 \overline{z_2})\mathbf{i_2}} = \\ &= \sqrt{x^2 - y^2 + 2\mathrm{Re}(-\mathbf{i_1} xy)} = \sqrt{x^2 - y^2} = |w|_{\mathbf{i_1}}. \end{aligned}$$

Finally,

$$|w|_{\mathbf{j}} = |x+y|\mathbf{e_1} + |x-y|\mathbf{e_2}\,.$$

Of course, the formula for $|w|_{\mathbf{i_1}} = |w|_{\mathbf{i_2}}$ reflects the hyperbolic structure of $\mathbb{R}^2$. More geometric reasonings will be given later.

5.4

Note an important relation between the hyperbolic conjugation and the quadratic form $\phi(x, y) := x^2 - y^2$; indeed,

$$x^2 - y^2 = (x + y\mathbf{j})(x - y\mathbf{j}) = (x + y\mathbf{j}) \cdot \overline{(x + y\mathbf{j})},$$

compare with the case of complex numbers. This leads immediately to a description of invertible and non-invertible hyperbolic numbers: for $w = x + y\mathbf{j}$ with $|w|_h \neq 0$

$$w \cdot \frac{\overline{w}}{|w|_h} = 1,$$

and denoting by $\mathbb{D}^{-1}$ the set of invertible elements, we obtain:

$$\mathbb{D}^{-1} = \{x + y\mathbf{j} \mid x^2 - y^2 \neq 0\}.$$

Comparing with Theorem 3.3.1, it is easily seen that $\mathbb{D}^{-1}$ is the intersection of $\mathbb{T}^{-1}$ and the plane (w_1, w_4) in the $\mathbb{R}^4 = \{(w_1, w_2, w_3, w_4)\}$. If now $\mathcal{O}_1$ denotes the set of zero divisors of $\mathbb{D}$, or its null-cone, then

$$\mathcal{NC}|_{\mathbb{D}} = \mathcal{O}_1 = \{x(1 \pm \mathbf{j}) \mid x \in \mathbb{R}\},$$

compare with Section 3.3 from which one can conclude that $\mathcal{O}_1$ is the intersection of $\mathcal{O}_2$ and the same plane (w_1, w_4).

5.5

It is instructive to extract the properties of the moduli of hyperbolic numbers from those presented in Section 4 for the general situation. We illustrate this with some examples.

5.5.1

Lemma 1.1.1 gives the following relation between the hyperbolic product and the real modulus: if $s = s_1 + s_2\mathbf{j}$, $t = t_1 + t_2\mathbf{j}$ then

$$|s \cdot t|^2 = |s|^2 \cdot |t|^2 + 4s_1s_2t_1t_2;$$

in particular,

$$|s \cdot t|^2 = |s|^2 \cdot |t|^2$$

if and only if $0 \in \{s_1, s_2, t_1, t_2\}$. What is more,

$$|s \cdot t| \leq \sqrt{2}|s||t|.$$

5.5.2

The analog of the Pythagoras Theorem (see Theorem 4.1.5) becomes:

$$x^2 + y^2 = \frac{1}{2}\left(|x+y|^2 + |x-y|^2\right)$$

which is trivial.

5.5.3

Since $|w|_h^2 = x^2 - y^2$ then $|w|_h = 0$ if and only if $w \in \mathcal{O}_1$. Some other properties of the hyperbolic modulus can be found in Section 4.2.8.

6 Bicomplex and hyperbolic numbers vs Clifford algebras

6.1

There are many ways of introducing the notion of a Clifford Algebra; we do not pretend to give here the most general and abstract description, we are going to explain just some basic ideas.

Let $\mathbf{u_1}, \ldots, \mathbf{u_n}$ be a basis of an n-dimensional real linear space $\mathcal{U}$, let us construct a real associative algebra $\mathcal{A}$ containing $\mathcal{U}$, in the following way: the basis elements will serve as imaginary units, i.e.

$$\begin{gathered}\mathbf{u_1}^2 = \cdots = \mathbf{u_k}^2 = -1, \\ \mathbf{u_{k+1}}^2 = \cdots = \mathbf{u_n}^2 = 1,\end{gathered}$$

for some integer k and anti-commute, i.e.

$$\mathbf{u_s}\mathbf{u_r} = -\mathbf{u_r}\mathbf{u_s}$$

for $s \neq r$. It is clear how this should be understood for $k = 0$ or $k = n$. Of course, for $s \neq r$, the product $\mathbf{u_s}\mathbf{u_r}$ is a new element, not from the set $\{\mathbf{u_1}, \ldots, \mathbf{u_n}\}$, the product $\mathbf{u_s}\mathbf{u_r}\mathbf{u_p} = (\mathbf{u_s}\mathbf{u_r})\mathbf{u_p} = \mathbf{u_s}(\mathbf{u_r}\mathbf{u_p})$ does not belong neither to $\{\mathbf{u_1}, \ldots, \mathbf{u_n}\}$ nor to $\{\mathbf{u_s}\mathbf{u_r} \mid \text{ any } s \neq r\}$. Finally, we arrive at the uniquely defined product of all the elements $\mathbf{u_1} \cdots \mathbf{u_n}$, in this order. Hence, any element of $\mathcal{A}$ is of the form

$$\begin{aligned} a &= a_0 + \sum_{k=1}^{n} a_k \mathbf{u_k} + \sum_{1 \le k < r \le n} a_{kr} \mathbf{u_k}\mathbf{u_r} \\ &+ \sum_{1 \le k < r < p \le n} a_{krp} \mathbf{u_k}\mathbf{u_r}\mathbf{u_p} + \cdots + a_{1,\ldots,n}\mathbf{u_1}\cdots\mathbf{u_n}, \end{aligned}$$

where all the coefficients are real numbers. There are many fine points here, and the interested reader can look for them in the books [BDS], [DSS], [GüSp], [Po], [Lo] and many others.

6.2

In a sense, Clifford algebras had been invented to factorize quadratic forms. Indeed, consider the quadratic form

$$x_0^2 + x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2$$

where k and n are as above, then there holds:

$$\begin{aligned} &x_0^2 + x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2 = \\ &= (x_0 + x_1\mathbf{u_1} + \cdots + x_k\mathbf{u_k} + x_{k+1}\mathbf{u_{k+1}} + \cdots + x_n\mathbf{u_n})\cdot \\ &\cdot(x_0 - x_1\mathbf{u_1} - \cdots x_k\mathbf{u_k} - x_{k+1}\mathbf{u_{k+1}} - \cdots - x_n\mathbf{u_n}) \,. \end{aligned}$$

6.3

Much in the same way one may consider $\{\mathbf{u_1}, \cdots, \mathbf{u_n}\}$ to be a basis of a complex linear, n-dimensional space which brings us to the general element of the form

$$c = c_0 + \sum_{k=1}^{n} c_k\mathbf{u_k} + \sum_{1\le k<r\le n} c_{kr}\mathbf{u_k}\mathbf{u_r} + \cdots + c_{1,\ldots,n}\mathbf{u_1}\cdots\mathbf{u_n}$$

where now all the coefficients are complex numbers $\alpha + \beta\mathbf{i}$ with the complex imaginary unit $\mathbf{i}$ commuting, by definition, with the Clifford imaginary units $\mathbf{u_1}, \ldots, \mathbf{u_n}$. Such a complex Clifford algebra $\mathcal{C}$ factorizes the complex quadratic form

$$\begin{aligned} &{z_0}^2 + {z_1}^2 + \cdots + z_k^2 - z_{k+1}^2 - \cdots - z_n^2 = \\ &= (z_0 + z_1\mathbf{u_1} + \cdots + z_k\mathbf{u_k} + \cdots + z_n\mathbf{u_n})\cdot \\ &\cdot(z_0 - z_1\mathbf{u_1} - \cdots z_k\mathbf{u_k} - \cdots - z_n\mathbf{u_n}). \end{aligned}$$

6.4

Consider some particular cases. Our approach leaves the field $\mathbb{R}$ outside the set of Clifford algebras although it is not hard to modify it and to include $\mathbb{R}$ also. Now, take $n = 1$, then we have two choices: $k = 0$ and $k = 1$. If $k = 1$ then $\mathbf{u_1^2} = -1$ and the arising *real* Clifford algebra is isomorphic to the field $\mathbb{C}$ of the *complex numbers*. If $k = 0$ then $\mathbf{u_1^2} = 1$, thus $\mathcal{A}$ coincides with $\mathbb{D}$, the set of hyperbolic numbers. Let us look at some complex Clifford algebras. Again, our approach does not include $\mathbb{C}$ into *complex* Clifford algebras but again one can easily modify it. Take now $k = 1$, i.e., $\mathbf{u_1^2} = -1$, then an arbitrary element of the algebra is

$$c_0 + c_1\mathbf{u_1}$$

where $c_0 = \alpha + \beta\mathbf{i}$, $c_1 = \gamma + \delta\mathbf{i}$, with $\mathbf{i}\mathbf{u_1} = \mathbf{u_1}\mathbf{i}$; thus we have the algebra isomorphic to the algebra $\mathbb{T}$ of bicomplex numbers. Taking $n = 1$, $k = 0$ one has: $\mathbf{u_1^2} = 1$ and $(\mathbf{i}\mathbf{u_1})^2 = -1 \cdot 1 = -1$, hence an arbitrary element $c_0 + c_1\mathbf{u_1}$

can be rewritten as $c_0 + c_1\mathbf{u_1} = \alpha + \beta\mathbf{i} + (\gamma + \delta\mathbf{i})\mathbf{u_1} = \alpha + \beta\mathbf{i} + \delta(\mathbf{iu_1}) + \gamma\mathbf{u_1}$ which, as we already know, is another form of writing a bicomplex number.

It is easy to see that $\mathbb{T}$ is a two-dimensional complex Clifford algebra which has $\mathbb{D}$ as its *real* (Clifford) subalgebra.

6.5

Note that $\mathbb{T}$ is singled-out from other complex Clifford algebras by the fact that all the others, i.e. Clifford algebras of complex dimension greater or equal to four, are non-commutative, meanwhile $\mathbb{C}$ and $\mathbb{D}$ are the only real commutative Clifford algebras.

6.6

Why "the only ones"? Let us show that for the equations $w^2 = 1$ and $w^2 = -1$, the set of all the solutions in $\mathbb{T}$ is $\{\pm 1; \pm\mathbf{i_1}; \pm\mathbf{i_2}; \pm\mathbf{j}\}$. Indeed, for $w = z_1 + z_2\mathbf{i_2}$ the equation $w^2 = 1$ is equivalent to

$$\begin{cases} z_1^2 - z_2^2 = 1, \\ z_1 z_2 = 0, \end{cases}$$

with z_1 and z_2 in $\mathbb{C}(\mathbf{i_1})$. The above system dissolves into the two independent ones:

$$\begin{cases} z_1 = 0, \\ z_2^2 = -1, \end{cases}$$

and

$$\begin{cases} z_2 = 0, \\ z_1^2 = 1, \end{cases}$$

and an easy analysis gives the solutions: $w = \pm\mathbf{i_1 i_2} = \pm\mathbf{j}$ for the first system and $w = \pm 1$ for the second.

Completely analogously, for the equation $w^2 = -1$ we obtain the solutions

$$w = \pm\mathbf{i_1} \quad \text{and} \quad w = \pm\mathbf{i_2}$$

which completes the proof.

6.7

The above means that $\mathbb{T}$ contains exactly three $\mathbb{C}$-like sets, i.e. the sets of the form $\mathbb{K} = \{x + y\mathbf{k} \mid x, y \in \mathbb{R}\}$ with $\mathbf{k}$ to be an element with the property $\mathbf{k}^2 = \pm 1$. Two of them, $\mathbb{C}(\mathbf{i_1})$ and $\mathbb{C}(\mathbf{i_2})$, are isomorphic to $\mathbb{C}$ and thus they are isomorphic; but being considered both inside $\mathbb{T}$ they cannot be identified; in this quality to be subsets of $\mathbb{T}$ they are essentially different. The third $\mathbb{C}$-like subset, $\mathbb{D}$, is quite different, at least it is not a field already, it has zero divisors.

Let $z = x + y\mathbf{k}$ be an element of $\mathbb{K}$, then the operation

$$z \in \mathbb{K} \mapsto \overline{z} := x - y\mathbf{k} \in \mathbb{K}$$

is a natural analog of the usual complex conjugation. It possesses the same fundamental property to factorize the corresponding quadratic form:

$$z \cdot \overline{z} = x^2 - \mathbf{k}^2 y^2 = \begin{cases} x^2 + y^2, & \text{for } \mathbf{k}^2 = -1, \\ x^2 - y^2, & \text{for } \mathbf{k}^2 = 1. \end{cases}$$

6.8

We introduced bicomplex numbers as a "duplication" of the set of complex numbers. This can be usefully seen in terms of tensor product of real linear spaces. Indeed, let us consider the tensor product of $\mathbb{R}^2$ with $\mathbb{C}$:

$$\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C}\,.$$

This means that $\mathbb{C}$ is seen, in fact, as $\mathbb{R}^2$ and that the resulting space is isomorphic to $\mathbb{R}^4$, or more exactly, to $\mathbb{R}^2 \times \mathbb{R}^2 := \{(z_1, z_2) \mid z_1, z_2 \in \mathbb{R}^2\}$. Now, let us identify $\mathbb{C}$ with $\mathbb{C}(\mathbf{i_2})$ and the first tensor factor, $\mathbb{R}^2$, with $\mathbb{C}(\mathbf{i_1})$ having in mind, of course, the additional structures in both copies of $\mathbb{R}^2$; under these identifications it is not an abuse to write

$$\mathbb{C}(\mathbf{i_1}) \otimes_{\mathbb{R}} \mathbb{C}(\mathbf{i_2}).$$

As an $\mathbb{R}$-linear space, it coincides already with $\mathbb{T}$, it suffices now to endow $\mathbb{C}(\mathbf{i_1}) \otimes_{\mathbb{R}} \mathbb{C}(\mathbf{i_2})$ with a multiplicative structure writing first $(z_1, z_2) \in \mathbb{C}(\mathbf{i_1}) \otimes_{\mathbb{R}} \mathbb{C}(\mathbf{i_2})$ as $z_1 + z_2\mathbf{i_2}$, and then defining the product of $(z_1 + z_2\mathbf{i_2})$ and $(z_3 + z_4\mathbf{i_2})$ to be

$$(z_1 + z_2\mathbf{i_2})(z_3 + z_4\mathbf{i_2}) := (z_1z_3 + z_2z_4) + (z_1z_4 + z_2z_3)\mathbf{i_2}.$$

Hence we may conclude that

$$\mathbb{T} = \mathbb{C}(\mathbf{i_1}) \otimes_{\mathbb{R}} \mathbb{C}(\mathbf{i_2}).$$

So, in a sense, $\mathbb{T}$ is a $\mathbb{C}(\mathbf{i_1})$-complexification of the set $\mathbb{C}(\mathbf{i_2})$ of "$\mathbf{i_2}$-complex" numbers.

6.8.1

In very similar ways $\mathbb{T}$ can be considered as tensor products of another pairs of two-dimensional algebras considered in Section 6.4. Seeing $\mathbb{R}^4$ as $\mathbb{R}^2 \times \mathbb{R}^2$ we identify now the first cartesian factor $\mathbb{R}^2$ with $\mathbb{C}(\mathbf{i_2})$ and the second one with $\mathbb{C}(\mathbf{i_1})$ thus arriving at the equality

$$\mathbb{T} = \mathbb{C}(\mathbf{i_2}) \otimes_{\mathbb{R}} \mathbb{C}(\mathbf{i_1}),$$

i.e. $w \in \mathbb{T}$ is represented as $w = a + b\mathbf{i_1}$ with a, b in $\mathbb{C}(\mathbf{i_2})$ and the multiplication in $\mathbb{T}$ is given by

$$(a + b\mathbf{i_1})(c + d\mathbf{i_1}) = (ac - bd) + (ad + bc)\mathbf{i_1} \, .$$

Since the idea, we guess, is clear already, we can enumerate another options. Identify subsequently one factor in $\mathbb{R}^2 \times \mathbb{R}^2$ with $\mathbb{D}$ and the other with $\mathbb{C}(\mathbf{i_1})$ or $\mathbb{C}(\mathbf{i_2})$ we obtain:

$$\begin{aligned}
\mathbb{T} &= \mathbb{D} \otimes_{\mathbb{R}} \mathbb{C}(\mathbf{i_1}), \\
\mathbb{T} &= \mathbb{D} \otimes_{\mathbb{R}} \mathbb{C}(\mathbf{i_2}), \\
\mathbb{T} &= \mathbb{C}(\mathbf{i_1}) \otimes_{\mathbb{R}} \mathbb{D}, \\
\mathbb{T} &= \mathbb{C}(\mathbf{i_2}) \otimes_{\mathbb{R}} \mathbb{D},
\end{aligned}$$

that is, $\mathbb{T}$ is a $\mathbb{D}$-extension (hyperbolization?) of the $\mathbb{C}(\mathbf{i_1})$ and $\mathbb{C}(\mathbf{i_2})$, but it is also $\mathbb{C}(\mathbf{i_1})$- and $\mathbb{C}(\mathbf{i_2})$-complexification of the set of hyperbolic numbers.

6.8.2

Let $w = w_1 + w_2\mathbf{i_1} + w_3\mathbf{i_2} + w_4\mathbf{j}$ be a bicomplex number written in a "real form". Re-arranging the summands and using the relation between $\mathbf{i_1}, \mathbf{i_2}, \mathbf{j}$ there is obtained:

- $w = (w_1 + w_2\mathbf{i_1}) + (w_3 + w_4\mathbf{i_1})\mathbf{i_2} =: z_1 + z_2\mathbf{i_2} \in \mathbb{T} = \mathbb{C}(\mathbf{i_1}) \otimes_{\mathbb{R}} \mathbb{C}(\mathbf{i_2})$ with $z_1 := w_1 + w_2\mathbf{i_1}$, $z_2 := w_3 + w_4\mathbf{i_1} \in \mathbb{C}(\mathbf{i_1})$.
- $w = (w_1 + w_3\mathbf{i_2}) + (w_2 + w_4\mathbf{i_2})\mathbf{i_1} =: a + b\mathbf{i_1} \in \mathbb{T} = \mathbb{C}(\mathbf{i_2}) \otimes_{\mathbb{R}} \mathbb{C}(\mathbf{i_1})$ with $a := w_1 + w_3\mathbf{i_2}$, $b = w_2 + w_4\mathbf{i_2} \in \mathbb{C}(\mathbf{i_2})$.
- $w = (w_1 + w_4\mathbf{j}) + (w_2 - w_3\mathbf{j})\mathbf{i_1} =: c + d\mathbf{i_1} \in \mathbb{T} = \mathbb{D} \otimes_{\mathbb{R}} \mathbb{C}(\mathbf{i_1})$ with $c = w_1 + w_4\mathbf{j}$, $d := w_2 - w_3\mathbf{j} \in \mathbb{D}$.
- $w = (w_1 + w_4\mathbf{j}) + (w_3 - w_2\mathbf{j})\mathbf{i_2} =: \alpha + \beta\mathbf{i_2} \in \mathbb{T} = \mathbb{D} \otimes_{\mathbb{R}} \mathbb{C}(\mathbf{i_2})$ with $\alpha = w_1 + w_2\mathbf{j}$, $\beta = w_3 - w_2\mathbf{j} \in \mathbb{D}$.
- w=$(w_1 + w_2\mathbf{i_1}) + (w_4 - w_2\mathbf{i_1})\mathbf{j} = \gamma + \delta\mathbf{j} \in \mathbb{T} = \mathbb{C}(\mathbf{i_1}) \otimes_{\mathbb{R}} \mathbb{D}$ with $\gamma = w_1 + w_2\mathbf{i_1}$, $\delta =: w_4 - w_3\mathbf{i_1} \in \mathbb{T} = \mathbb{C}(\mathbf{i_1})$.
- $w = (w_1 + w_3\mathbf{i_2}) + (w_4 - w_2\mathbf{i_2})\mathbf{j} =: u + v\mathbf{j} \in \mathbb{T} = \mathbb{C}(\mathbf{i_2}) \otimes_{\mathbb{R}} \mathbb{D}$ with $u = w_1 + w_3\mathbf{i_2}$, $v := w_4 - w_2\mathbf{i_2} \in \mathbb{C}(\mathbf{i_2})$.

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