The Inverse Iteration Method for Julia Sets in the 3-Dimensional Space

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The 8th Chaotic Modeling and Simulation International Conference
Paris, France
26-29 May 2015

[1]Research supported by CRSNG (Canada).

- Fractal sets created by iterative processes have been greatly studied in the past decades. After being displayed in the complex plane, they became part of the 3-dimensional space when Norton gave straightforward algorithms using iteration with quaternions. However, as established by Bedding and Briggs, it seems that no interesting dynamics could arise from this approach based on the local rotations of the classical sets. Another set of numbers revealed to be possibly more appropriate: **Bicomplex Numbers**.
- Bicomplex numbers, just like quaternions, are a generalization of complex numbers by means of entities specified by four real numbers. These two number systems, however, are different in two important ways: quaternions, which form a division algebra, are noncommutative, whereas bicomplex numbers are commutative but do not form a division algebra.

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- Bicomplex numbers, just like quaternions, are a generalization of complex numbers by means of entities specified by four real numbers. These two number systems, however, are different in two important ways: quaternions, which form a division algebra, are noncommutative, whereas bicomplex numbers are commutative but do not form a division algebra.

Definition

Bicomplex numbers are defined as

$$\mathbb{BC} := \{z_1 + z_2\mathbf{i_2} \mid z_1, z_2 \in \mathbb{C}(\mathbf{i_1})\}$$

where the imaginary units $\mathbf{i_1}, \mathbf{i_2}$ and $\mathbf{j}$ are governed by the rules: $\mathbf{i_1}^2 = \mathbf{i_2}^2 = -1$, $\mathbf{j}^2 = 1$ and

$$\begin{array}{rcccr} \mathbf{i_1}\mathbf{i_2} & = & \mathbf{i_2}\mathbf{i_1} & = & \mathbf{j}, \\ \mathbf{i_1}\mathbf{j} & = & \mathbf{j}\mathbf{i_1} & = & -\mathbf{i_2}, \\ \mathbf{i_2}\mathbf{j} & = & \mathbf{j}\mathbf{i_2} & = & -\mathbf{i_1}. \end{array}$$

- Note that we define $\mathbb{C}(\mathbf{i}_k) := \{x + y\mathbf{i}_k \mid \mathbf{i}_k^2 = -1 \text{ and } x, y \in \mathbb{R}\}$ for $k = 1, 2$.

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In fact, the bicomplex numbers

$$\mathbb{BC} \cong \mathrm{Cl}_{\mathbb{C}}(1,0) \cong \mathrm{Cl}_{\mathbb{C}}(0,1)$$

are *unique* among the complex Clifford algebras in the sense that they are commutative but not division algebra. It is also convenient to write the set of bicomplex numbers as

$$\mathbb{BC} := \{w_0 + w_1\mathbf{i_1} + w_2\mathbf{i_2} + w_3\mathbf{j} \mid w_0, w_1, w_2, w_3 \in \mathbb{R}\}.$$

Zero divisors make up the so-called null cone $\mathcal{NC}$. That terminology comes from the fact that when w is written as $z_1 + z_2\mathbf{i_2}$, zero divisors are such that $z_1^2 + z_2^2 = 0$.

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- It is also important to know that every bicomplex number $w = z_1 + z_2\mathbf{i_2}$ has the following unique idempotent representation:

$$\begin{aligned} z_1 + z_2\mathbf{i_2} &= \mathcal{P}_1(w)\mathbf{e_1} + \mathcal{P}_2(w)\mathbf{e_2} \\ &:= (z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2} \end{aligned}$$

  where $\mathbf{e_1} = \frac{1+\mathbf{j}}{2}$ and $\mathbf{e_2} = \frac{1-\mathbf{j}}{2}$.

- From this, we can introduce the bicomplex cartesian product:

Definition

We say that $X \subseteq \mathbb{BC}$ is a $\mathbb{BC}$-cartesian set determined by X_1 and X_2 if $X = X_1 \times_e X_2$ where

$$X_1 \times_e X_2 := \{z_1 + z_2\mathbf{i_2} \in \mathbb{BC} : z_1 + z_2\mathbf{i_2} = w_1\mathbf{e_1} + w_2\mathbf{e_2}, (w_1, w_2) \in X_1 \times X_2\}.$$

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Let $\mathcal{K}_c = \{z \in \mathbb{C} \mid \{P_c^n(z)\}_{n=0}^{\infty} \text{ is bounded } \}$ be the filled-in Julia set associated to $P_c = z^2 + c$. The Julia set related to P_c is denoted by $\mathcal{J}_c$ and defined as either one of the following:

1. The boundary of the filled-in Julia set: $\mathcal{J}_c = \partial\mathcal{K}_c$;
2. The set of points $z \in \mathbb{C}$ for which the forward iterates do not form a normal family at z.

The second definition leads to the following theorem that justifies the inverse iteration method.

Theorem

Let P be a monic complex polynomial of degree $d \geq 2$. For any $z_1 \in \mathcal{J}_P$, the set of inverse iterates $\left\{ \bigcup_{k=k_1}^{\infty} P^{-k}(z_1) \right\}$ is dense in $\mathcal{J}_P$ for all whole number $k_1 \geq 1$.

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To generate and display $\mathcal{J}_c$ in the complex plane, it suffices to take $z_1 \in \mathcal{J}_c$ and compute its inverse iteratively, up to a maximum number of iterations. Since the inverse is given by the complex square root function $\sqrt{z-c}$ which is multivalued, two different approaches may be used. The first one is to compute all branches of the inverse at every iteration, leading to a great number of points generated. The second option is to randomly choose one of the branches of the inverse at each iteration and compute only this one. This last approach seems more appropriate for it is faster and requires less memory space.

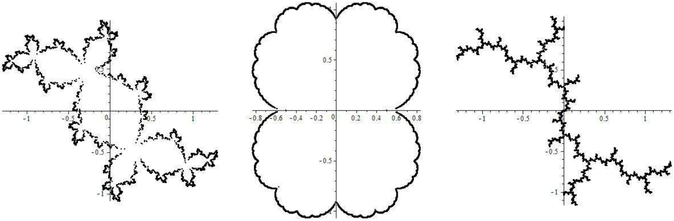


Figure: Julia sets of the complex plane

1. Introduction
2. Preliminaries
   - Bicomplex numbers
   - Julia sets in the complex plane
3. Characterization of Bicomplex Julia Sets
   - Boundary of bicomplex cartesian sets
   - Normal families
4. Bicomplex Inverse Iteration Method

We now introduce basic results in **bicomplex dynamics**.

Definition

The bicomplex filled-in Julia set corresponding to $c \in \mathbb{BC}$ is defined as

$$\mathcal{K}_{2,c} = \{w \in \mathbb{BC} \mid \{P_c^n(w)\}_{n=0}^{\infty} \text{ is bounded}\}.$$

This set corresponds to a particular bicomplex cartesian set determined by the idempotent components of c.

Theorem

Let $c = c_1 + c_2\mathbf{i_2} \in \mathbb{BC}$. *Then* $\mathcal{K}_{2,c} = \mathcal{K}_{c_1 - c_2\mathbf{i_1}} \times_e \mathcal{K}_{c_1 + c_2\mathbf{i_1}}$.

Definition

The bicomplex Julia set corresponding to $c \in \mathbb{BC}$ is the boundary of the associated filled-in Julia set: $\mathcal{J}_{2,c} = \partial\mathcal{K}_{2,c}$.

The next result leads to a useful point of view on bicomplex Julia sets.

Theorem

Let $X_1, X_2 \subseteq \mathbb{C}(\mathbf{i_1})$ be nonempty and $X \subseteq \mathbb{BC}$ such that $X = X_1 \times_e X_2$. The boundary of X is the union of three bicomplex cartesian sets:

$$\partial X = (\partial X_1 \times_e X_2) \cup (X_1 \times_e \partial X_2) \cup (\partial X_1 \times_e \partial X_2).$$

This theorem allows a better understanding of the structure of bicomplex Julia sets, as given in the next corollary. We see that the set is solely determined by complex Julia sets and filled-in Julia sets associated to the idempotent components of c.

Corollary

The bicomplex Julia set corresponding to $c = c_1 + c_2\mathbf{i_2} \in \mathbb{BC}$ may be expressed as

$$\mathcal{J}_{2,c} = (\mathcal{J}_{c_1 - c_2\mathbf{i_1}} \times_e \mathcal{K}_{c_1 + c_2\mathbf{i_1}}) \cup (\mathcal{K}_{c_1 - c_2\mathbf{i_1}} \times_e \mathcal{J}_{c_1 + c_2\mathbf{i_1}})$$

where $\mathcal{J}_{c_1 - c_2\mathbf{i_1}} \times_e \mathcal{J}_{c_1 + c_2\mathbf{i_1}} \subseteq \mathcal{J}_{2,c}$.

This corollary provides a useful insight on bicomplex Julia sets that leads to an easy display in the usual 3D space. Indeed, for a certain $c = c_1 + c_2\mathbf{i_2} \in \mathbb{BC}$, it suffices to generate points of $\mathcal{J}_{c_1 - c_2\mathbf{i_1}}$, $\mathcal{J}_{c_1 + c_2\mathbf{i_1}}$, $\mathcal{K}_{c_1 - c_2\mathbf{i_1}}$ and $\mathcal{K}_{c_1 + c_2\mathbf{i_1}}$ and create the bicomplex cartesian sets of the characterization using the idempotent representation. Rewriting the elements of the set under the representation $a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j}$, a 3D cut is applied by keeping only the points for which the absolute value of d (the $\mathbf{j}$-component) is less than a certain value $\epsilon > 0$. The display of the other components in the 3D space produces an approximation of $\mathcal{J}_{2,c}$ that we may observe.

Two examples are given in the next figures for

$$c = 0,25 \text{ and } c = 0,0635 + 0,3725\mathbf{i_1} + 0,3725\mathbf{i_2} + 0,1865\mathbf{j}.$$

In the case of $c = 0,0635 + 0,3725\mathbf{i_1} + 0,3725\mathbf{i_2} + 0,1865\mathbf{j}$, we have $c_1 - c_2\mathbf{i_1} = 0,25$ and $c_1 + c_2\mathbf{i_1} = -0,123 + 0,745\mathbf{i_1}$. Both complex shapes may be seen when moving the object around. The color black represents the set $\mathcal{J}_{c_1 - c_2\mathbf{i_1}} \times_e \mathcal{J}_{c_1 + c_2\mathbf{i_1}}$ (the one determined by complex Julia sets) and provides the main structure of the set.

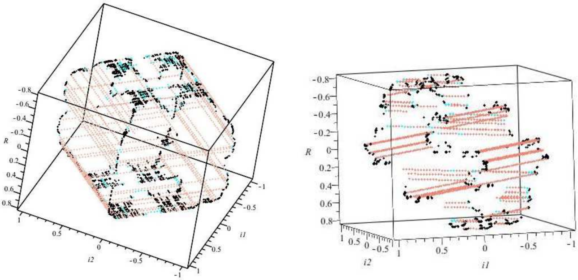


Figure: Bicomplex Julia sets in 3D space

As in one complex variable, another way to define bicomplex Julia sets is to use the concept of normal families. Let us first recall the following definition of bicomplex normality.

Definition

The family $\boldsymbol{F}$ of bicomplex holomorphic functions defined on a domain $D \subseteq \mathbb{BC}$ is said to be **normal** in D if every sequence in $\boldsymbol{F}$ contains a subsequence which on compact subsets of D either: 1. converges uniformly to a limit function or 2. converges uniformly to ∞. The family $\boldsymbol{F}$ is said to be **normal at a point** $z \in D$ if it is normal in some neighborhood of z in D.

We say that the sequence $\{w_n\}$ of bicomplex numbers converges to ∞ if and only if the norm $\{\|w_n\|\}$ congerges to ∞.

With this definition we have this other possible definition of bicomplex Julia sets.

Definition

Let $P(w)$ be a bicomplex polynomial. We define the bicomplex Julia set for P as

$$\mathcal{J}_2(P) = \{w \in \mathbb{BC} \mid \{P^{\circ n}(w)\} \text{ is not normal}\}.$$

In the particular case of the bicomplex holomorphic polynomial $P_c(w) = w^2 + c$ where $\boldsymbol{F} := \{P_c^{\circ n}, \ \forall n \in \mathbb{N}\}$, we have this following characterization of bicomplex Julia sets.

Corollary

Let $P_c(w) = w^2 + c$, then

$$\mathcal{J}_2(P_c) = \partial\mathcal{K}_{2,c} = \mathcal{J}_{2,c}.$$

To extend the inverse iteration method for bicomplex Julia sets, we need to find a result similar to the complex case for $\mathcal{J}_{2,c}$. However, the structure of this set suggests that it is impossible to do so. Indeed, the set of inverse iterates of P_c cannot be dense in $\mathcal{J}_{2,c}$ due to the presence of $\mathcal{K}_{c_1-c_2\mathbf{i_1}}$ and $\mathcal{K}_{c_1+c_2\mathbf{i_1}}$ (where the complex inverse iterates are not usually dense). Nevertheless, it appears relevant to adapt the inverse iteration method for $\mathcal{J}_{c_1-c_2\mathbf{i_1}} \times_e \mathcal{J}_{c_1+c_2\mathbf{i_1}}$ since this specific case coincide with this specific condition: $z_1 + z_2\mathbf{i_2} \in \mathcal{J}_{2,c}$ such that

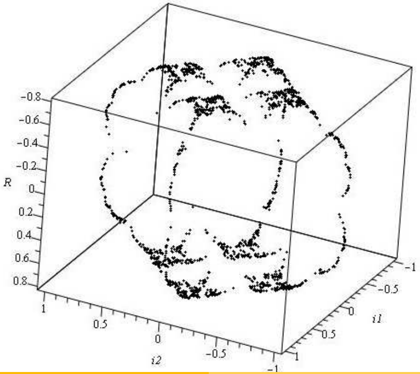
$\{[\mathcal{P}_1(w^2+c)]^{\circ n}(z_1-z_2\mathbf{i_1})\}$ and $\{[\mathcal{P}_2(w^2+c)]^{\circ n}(z_1+z_2\mathbf{i_1})\}$ are not normal.

Here is the main result of this talk:

Theorem

Let $c = c_1 + c_2\mathbf{i_2} \in \mathbb{BC}$ and w_1 a fixed point of P_c such that $w_1 \in \mathcal{J}_{c_1-c_2\mathbf{i_1}} \times_e \mathcal{J}_{c_1+c_2\mathbf{i_1}}$. The set $\left\{ \bigcup_{k=1}^{\infty} P_c^{-k}(w_1) \right\}$ is dense in $\mathcal{J}_{c_1-c_2\mathbf{i_1}} \times_e \mathcal{J}_{c_1+c_2\mathbf{i_1}}$.

Consequently, part of a bicomplex Julia set may be created using an adapted version of the inverse iteration method. As an example, for $c = 0,25$ the set $\mathcal{J}_{c_1 - c_2\mathbf{i_1}} \times_e \mathcal{J}_{c_1 + c_2\mathbf{i_1}}$ is generated and given in this figure. At each iteration, the inverse is computed choosing randomly one of the foor branches of the bicomplex square root function $\sqrt{w - c}$. A 3D cut is applied as before, by keeping only the points for which the absolute value of the $\mathbf{j}$-component is less than a certain value $\epsilon > 0$.



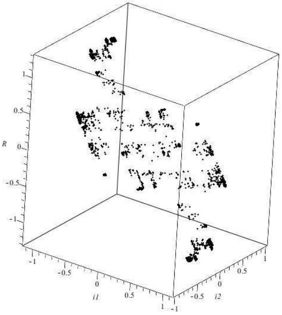


Figure: $\mathcal{J}_{2,c}$ for $c = \mathbf{i_1}$ in 3D space

Consider $c = \mathbf{i_1}$, that is $c = c_1 + c_2\mathbf{i_2} \in \mathbb{BC}$ with $c_1 = \mathbf{i_1}$ and $c_2 = 0$. Then $c_1 - c_2\mathbf{i_1} = c_1 + c_2\mathbf{i_1} = \mathbf{i_1}$ and $\mathcal{K}_{2,c} = \mathcal{J}_{2,c} = \mathcal{J}_{\mathbf{i_1}} \times_e \mathcal{J}_{\mathbf{i_1}}$ is a dendrite. Again, the inverse iteration method is adaptable to generate this set and display it in the 3D space.

Conclusion

Since the bicomplex polynomial $P_c(w) = w^2 + c$ is the following mapping of $\mathbb{C}^2$: $\left(z_1^2 - z_2^2 + c_1,\ 2z_1z_2 + c_2\right)$ where $w = z_1 + z_2\mathbf{i_2} := (z_1, z_2)$ and $c = c_1 + c_2\mathbf{i_2} := (c_1, c_2)$, bicomplex dynamics is a particular case of dynamics of several complex variables. This lead to the first generalization of the inverse iteration method in two complex variables.

Thank you!

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