

Bicomplex Quantum Mechanics: II. The Hilbert Space

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Abstract. Using the bicomplex numbers $\mathbb{T} \cong \text{Cl}_{\mathbb{C}}(1, 0) \cong \text{Cl}_{\mathbb{C}}(0, 1)$ which is a commutative ring with zero divisors defined by $\mathbb{T} = \{w_0 + w_1 \mathbf{i}_1 + w_2 \mathbf{i}_2 + w_3 \mathbf{j} \mid w_0, w_1, w_2, w_3 \in \mathbb{R}\}$ where $\mathbf{i}_1^2 = -1$, $\mathbf{i}_2^2 = -1$, $\mathbf{j}^2 = 1$ and $\mathbf{i}_1 \mathbf{i}_2 = \mathbf{j} = \mathbf{i}_2 \mathbf{i}_1$, we construct hyperbolic and bicomplex Hilbert spaces. Linear functionals and dual spaces are considered on these spaces and properties of linear operators are obtained; in particular it is established that the eigenvalues of a bicomplex self-adjoint operator are in the set of hyperbolic numbers.

Keywords. Bicomplex numbers, hyperbolic numbers, complex Clifford algebras, generalized quantum mechanics, Hilbert spaces, free modules, linear functionals, self-adjoint operators.

1. Introduction

Many papers have been written on the extension of the formalism of quantum mechanics. These generalizations have been done mainly over quaternions or over the Cayley algebra (octonions), see for instance [1, 2, 3, 4]. The reason why people have worked mainly on this algebraic structures to generalize quantum mechanics comes from the fact that there exist only four normed division algebras [5]: reals (\mathbb{R}), complex numbers (\mathbb{C}), quaternions (\mathbb{H}) and the Cayley algebra (\mathbb{O}). The Cayley algebra has an important blank since associativity is crucial. Indeed, in [1] it is shown that quantum mechanics cannot be formulated over the Cayley algebra since, in at least two instances, associativity is needed for the existence of Hilbert space. Quantum mechanics over quaternions seems to work better [1, 2, 3, 6]. However, recently some interest has been deployed to study quantum mechanics for associative and commutative algebras beyond the paradigm of algebras without zero divisors [7, 8, 9]. This leads to a wide spectrum of possibilities, among which we have the hyperbolic numbers $\mathbb{D} \cong \text{Cl}_{\mathbb{R}}(0, 1)$ (also called duplex numbers) [10], the bicomplex numbers $\mathbb{T} \cong \text{Cl}_{\mathbb{C}}(1, 0) \cong \text{Cl}_{\mathbb{C}}(0, 1)$ [11] and, more generally, the multicomplex numbers [12, 13].

In recent years, the theory of bicomplex numbers and bicomplex functions has found many applications, see for instance [14, 15, 16, 17, 18]. Bicomplex numbers are a commutative ring with unity which contains the field of complex numbers and the commutative ring of hyperbolic numbers. Bicomplex (hyperbolic) numbers are *unique* among the complex (real) Clifford algebras in that they are commutative but not division algebras. In fact, bicomplex numbers generalize (complexify) hyperbolic numbers. Note that Hilbert spaces over hyperbolic numbers that have been studied in [8, 9] and [19] are different from the hyperbolic Hilbert space that we consider in this paper.

In Section 2 we give an overview of the fundamental theory of bicomplex analysis necessary for this article. Section 3 is devoted to free modules over the ring of bicomplex numbers (which is not a C^* -algebra). A fundamental result useful for the rest of the paper is presented: the unique decomposition of any elements of our free module M into two elements of a standard (complex) vector space in terms of the idempotent basis. The Section 4 (and 5) introduces the bicomplex scalar product (the hyperbolic scalar product). In particular, it is shown that one can construct a metric space from M and our bicomplex scalar product. In Section 6, we define the bicomplex Hilbert space; two examples are given. Section 7 introduces the dual space M^* and re-examines the previous Sections in terms of the Dirac notation. Finally, Section 8 concerns linear operators or more specifically adjoint and self-adjoint operators as well as the bicomplex eigenvectors equation.

2. Preliminaries

Bicomplex numbers are defined as [11, 12, 20]

$$\mathbb{T} := \{z_1 + z_2\mathbf{i}_2 \mid z_1, z_2 \in \mathbb{C}(\mathbf{i}_1)\}, \quad (2.1)$$

where the imaginary units $\mathbf{i}_1, \mathbf{i}_2$ and \mathbf{j} are governed by the rules: $\mathbf{i}_1^2 = \mathbf{i}_2^2 = -1$, $\mathbf{j}^2 = 1$ and

$$\begin{aligned} \mathbf{i}_1\mathbf{i}_2 &= \mathbf{i}_2\mathbf{i}_1 = \mathbf{j}, \\ \mathbf{i}_1\mathbf{j} &= \mathbf{j}\mathbf{i}_1 = -\mathbf{i}_2, \\ \mathbf{i}_2\mathbf{j} &= \mathbf{j}\mathbf{i}_2 = -\mathbf{i}_1, \end{aligned} \quad (2.2)$$

where we define $\mathbb{C}(\mathbf{i}_k) := \{x + y\mathbf{i}_k \mid \mathbf{i}_k^2 = -1 \text{ and } x, y \in \mathbb{R}\}$ for $k = 1, 2$. Hence it is easy to see that the multiplication of two bicomplex numbers is commutative.

It is also convenient to write the set of bicomplex numbers as

$$\mathbb{T} := \{w_0 + w_1\mathbf{i}_1 + w_2\mathbf{i}_2 + w_3\mathbf{j} \mid w_0, w_1, w_2, w_3 \in \mathbb{R}\}. \quad (2.3)$$

In particular, in equation (2.1), if we put $z_1 = x$ and $z_2 = y\mathbf{i}_1$ with $x, y \in \mathbb{R}$, then we obtain the subalgebra of hyperbolic numbers: $\mathbb{D} = \{x + y\mathbf{j} \mid \mathbf{j}^2 = 1, x, y \in \mathbb{R}\}$.

Complex conjugation plays an important role both for algebraic and geometric properties of \mathbb{C} , as well as in standard quantum mechanics. For bicomplex

numbers, there are three possible conjugations. Let $w \in \mathbb{T}$ and $z_1, z_2 \in \mathbb{C}(\mathbf{i}_1)$ such that $w = z_1 + z_2 \mathbf{i}_2$. Then we define the three conjugations as:

$$w^{\dagger_1} = (z_1 + z_2 \mathbf{i}_2)^{\dagger_1} := \bar{z}_1 + \bar{z}_2 \mathbf{i}_2, \quad (2.4a)$$

$$w^{\dagger_2} = (z_1 + z_2 \mathbf{i}_2)^{\dagger_2} := z_1 - z_2 \mathbf{i}_2, \quad (2.4b)$$

$$w^{\dagger_3} = (z_1 + z_2 \mathbf{i}_2)^{\dagger_3} := \bar{z}_1 - \bar{z}_2 \mathbf{i}_2, \quad (2.4c)$$

where \bar{z}_k is the standard complex conjugate of complex numbers $z_k \in \mathbb{C}(\mathbf{i}_1)$. If we say that the bicomplex number $w = z_1 + z_2 \mathbf{i}_2 = w_0 + w_1 \mathbf{i}_1 + w_2 \mathbf{i}_2 + w_3 \mathbf{j}$ has the ‘signature’ $(+++)$, then the conjugations of type 1,2 or 3 of w have, respectively, the signatures $(+ - + -)$, $(+ + - -)$ and $(+ - - +)$. We can verify easily that the composition of the conjugates gives the four-dimensional abelian Klein group:

\circ	\dagger_0	\dagger_1	\dagger_2	\dagger_3
\dagger_0	\dagger_0	\dagger_1	\dagger_2	\dagger_3
\dagger_1	\dagger_1	\dagger_0	\dagger_3	\dagger_2
\dagger_2	\dagger_2	\dagger_3	\dagger_0	\dagger_1
\dagger_3	\dagger_3	\dagger_2	\dagger_1	\dagger_0

(2.5)

where $w^{\dagger_0} := w \ \forall w \in \mathbb{T}$.

All of the three kinds of conjugation have some of the standard properties of conjugations, such as:

$$(s + t)^{\dagger_k} = s^{\dagger_k} + t^{\dagger_k}, \quad (2.6)$$

$$(s^{\dagger_k})^{\dagger_k} = s, \quad (2.7)$$

$$(s \cdot t)^{\dagger_k} = s^{\dagger_k} \cdot t^{\dagger_k}, \quad (2.8)$$

for $s, t \in \mathbb{T}$ and $k = 0, 1, 2, 3$.

We know that the product of a standard complex number with its conjugate gives the square of the Euclidean metric in \mathbb{R}^2 . The analogs of this, for bicomplex numbers, are the following. Let $z_1, z_2 \in \mathbb{C}(\mathbf{i}_2)$ and $w = z_1 + z_2 \mathbf{i}_2 \in \mathbb{T}$, then we have that [11]:

$$|w|_{\mathbf{i}_1}^2 := w \cdot w^{\dagger_2} = z_1^2 + z_2^2 \in \mathbb{C}(\mathbf{i}_1), \quad (2.9a)$$

$$|w|_{\mathbf{i}_2}^2 := w \cdot w^{\dagger_1} = (|z_1|^2 - |z_2|^2) + 2\text{Re}(z_1 \bar{z}_2) \mathbf{i}_2 \in \mathbb{C}(\mathbf{i}_2), \quad (2.9b)$$

$$|w|_{\mathbf{j}}^2 := w \cdot w^{\dagger_3} = (|z_1|^2 + |z_2|^2) - 2\text{Im}(z_1 \bar{z}_2) \mathbf{j} \in \mathbb{D}, \quad (2.9c)$$

where the subscript of the square modulus refers to the subalgebra $\mathbb{C}(\mathbf{i}_1)$, $\mathbb{C}(\mathbf{i}_2)$ or \mathbb{D} of \mathbb{T} in which w is projected.

Note that for $z_1, z_2 \in \mathbb{C}(\mathbf{i}_1)$ and $w = z_1 + z_2 \mathbf{i}_2 \in \mathbb{T}$, we can define the usual (Euclidean in \mathbb{R}^4) norm of w as $|w| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\text{Re}(|w|_{\mathbf{j}}^2)}$.

It is easy to verify that $w \cdot \frac{w^{\dagger 2}}{|w|_{\mathbf{i}_1}^2} = 1$. Hence, the inverse of w is given by

$$w^{-1} = \frac{w^{\dagger 2}}{|w|_{\mathbf{i}_1}^2}. \quad (2.10)$$

From this, we find that the set \mathcal{NC} of zero divisors of \mathbb{T} , called the *null-cone*, is given by $\{z_1 + z_2 \mathbf{i}_2 \mid z_1^2 + z_2^2 = 0\}$, which can be rewritten as

$$\mathcal{NC} = \{z(\mathbf{i}_1 \pm \mathbf{i}_2) \mid z \in \mathbb{C}(\mathbf{i}_1)\}. \quad (2.11)$$

Let us also now recall the following three *real moduli* (see [11] and [20]):

- 1) For $s, t \in \mathbb{T}$, we define the first modulus as $|\cdot|_{\mathbf{1}} := \|\cdot\|_{\mathbf{i}_1}$. This modulus has the following properties:
 - a) $|\cdot|_{\mathbf{1}} : \mathbb{T} \rightarrow \mathbb{R}$,
 - b) $|s|_{\mathbf{1}} \geq 0$ with $|s|_{\mathbf{1}} = 0$ iff $s \in \mathcal{NC}$,
 - c) $|s \cdot t|_{\mathbf{1}} = |s|_{\mathbf{1}} \cdot |t|_{\mathbf{1}}$.

From this definition, we can rewrite this real pseudo-modulus in a much practical way as

$$|w|_{\mathbf{1}} = |z_1^2 + z_2^2|^{1/2}$$

or

$$|w|_{\mathbf{1}} = \sqrt[4]{ww^{\dagger 1}w^{\dagger 2}w^{\dagger 3}}.$$

- 2) For $s, t \in \mathbb{T}$, we can define formally the second real modulus as $|\cdot|_{\mathbf{2}} := \|\cdot\|_{\mathbf{i}_2}$. But an easy computation leads to

$$|w|_{\mathbf{2}} = |w|_{\mathbf{1}} = |z_1^2 + z_2^2|^{1/2}, \quad (2.12)$$

meaning that there are no reasons to introduce $|\cdot|_{\mathbf{2}}$

- 3) One more option is to define the third modulus as $|\cdot|_{\mathbf{3}} := \|\cdot\|_{\mathbf{j}}$. It has the following properties:
 - a) $|\cdot|_{\mathbf{3}} : \mathbb{T} \rightarrow \mathbb{R}$,
 - b) $|s|_{\mathbf{3}} \geq 0$ with $|s|_{\mathbf{3}} = 0$ iff $s = 0$,
 - c) $|s + t|_{\mathbf{3}} \leq |s|_{\mathbf{3}} + |t|_{\mathbf{3}}$,
 - d) $|s \cdot t|_{\mathbf{3}} \leq \sqrt{2}|s|_{\mathbf{3}} \cdot |t|_{\mathbf{3}}$.
 - e) $|\lambda \cdot t|_{\mathbf{3}} = |\lambda| \cdot |t|_{\mathbf{3}}$, for $\lambda \in \mathbb{C}(\mathbf{i}_1)$ or $\mathbb{C}(\mathbf{i}_2)$.

Hence $|\cdot|_{\mathbf{3}}$ determines a structure of a real normed algebra on \mathbb{T} . What is more, one gets directly that

$$|w|_{\mathbf{3}} = \sqrt{|z_1|^2 + |z_2|^2}, \quad (2.13)$$

for $w = z_1 + z_2 \mathbf{i}_2$ with $z_1, z_2 \in \mathbb{C}(\mathbf{i}_1)$, i.e., in fact this is just the Euclidean metric in \mathbb{R}^4 written in a form compatible with the multiplicative structure of bicomplex numbers.

Note also that

- (i) $|w|_{\mathbf{j}} = |z_1 - z_2 \mathbf{i}_1|_{\mathbf{e}_1} + |z_1 + z_2 \mathbf{i}_1|_{\mathbf{e}_2} \in \mathbb{D}$, $\forall w = z_1 + z_2 \mathbf{i}_2 \in \mathbb{T}$,
- (ii) $|s \cdot t|_{\mathbf{j}} = |s|_{\mathbf{j}} |t|_{\mathbf{j}}$ $\forall s, t \in \mathbb{T}$.

Finally, let us mention that any bicomplex numbers can be written using an orthogonal idempotent basis defined by

$$\mathbf{e}_1 = \frac{1+\mathbf{j}}{2} \quad \text{and} \quad \mathbf{e}_2 = \frac{1-\mathbf{j}}{2},$$

where $\mathbf{e}_1^2 = \mathbf{e}_1$, $\mathbf{e}_2^2 = \mathbf{e}_2$, $\mathbf{e}_1 + \mathbf{e}_2 = 1$ and $\mathbf{e}_1\mathbf{e}_2 = 0 = \mathbf{e}_2\mathbf{e}_1$. Indeed, it is easy to show that for any $z_1 + z_2\mathbf{i}_2 \in \mathbb{T}$, $z_1, z_2 \in \mathbb{C}(\mathbf{i}_1)$, we have

$$z_1 + z_2\mathbf{i}_2 = (z_1 - z_2\mathbf{i}_1)\mathbf{e}_1 + (z_1 + z_2\mathbf{i}_1)\mathbf{e}_2. \quad (2.14)$$

3. \mathbb{T} -Module

The set of bicomplex numbers is a commutative ring. So, to define a kind of vector space over \mathbb{T} , we have to deal with the algebraic concept of modules. We denote by M a free \mathbb{T} -module with the finite \mathbb{T} -basis $\{\hat{m}_l \mid l \in \{1, \dots, n\}\}$. Hence,

$$M = \left\{ \sum_{l=1}^n x_l \hat{m}_l \mid x_l \in \mathbb{T} \right\}.$$

Let us now define

$$V := \left\{ \sum_{l=1}^n x_l \hat{m}_l \mid x_l \in \mathbb{C}(\mathbf{i}_1) \right\} \subset M. \quad (3.1)$$

The set V is a free $\mathbb{C}(\mathbf{i}_1)$ -module which depends on a given \mathbb{T} -basis of M . In fact, V is a complex vector space of dimension n with the basis $\{\hat{m}_l \mid l \in \{1, \dots, n\}\}$. For a complete treatment of Module Theory, see [21].

Theorem 1. Let $\hat{X} = \sum_{l=1}^n x_l \hat{m}_l$, $x_l \in \mathbb{T}$, for all $l \in \{1, \dots, n\}$. Then, there exist $\hat{X}_{\mathbf{e}_1}, \hat{X}_{\mathbf{e}_2} \in V$ such that

$$\hat{X} = \mathbf{e}_1 \hat{X}_{\mathbf{e}_1} + \mathbf{e}_2 \hat{X}_{\mathbf{e}_2}.$$

Proof. From equation (2.14), it is always possible to decompose a bicomplex number in term of the idempotent basis. So let us write $x_l = x_{1l}\mathbf{e}_1 + x_{2l}\mathbf{e}_2$ where $x_{1l}, x_{2l} \in \mathbb{C}(\mathbf{i}_1)$, for all $l \in \{1, \dots, n\}$. Hence,

$$\begin{aligned} \hat{X} &= \sum_{l=1}^n x_l \hat{m}_l = \sum_{l=1}^n (x_{1l}\mathbf{e}_1 + x_{2l}\mathbf{e}_2) \hat{m}_l = \mathbf{e}_1 \sum_{l=1}^n (x_{1l} \hat{m}_l) + \mathbf{e}_2 \sum_{l=1}^n (x_{2l} \hat{m}_l) \\ &= \mathbf{e}_1 \hat{X}_{\mathbf{e}_1} + \mathbf{e}_2 \hat{X}_{\mathbf{e}_2} \end{aligned}$$

where $\hat{X}_{\mathbf{e}_k} := \sum_{l=1}^n (x_{kl} \hat{m}_l)$ for $k = 1, 2$.

Corollary 1. *The elements $\widehat{X}_{\mathbf{e}_1}$ and $\widehat{X}_{\mathbf{e}_2}$ are uniquely determined. In other words, $\mathbf{e}_1\widehat{X}_{\mathbf{e}_1} + \mathbf{e}_2\widehat{X}_{\mathbf{e}_2} = \mathbf{e}_1\widehat{Y}_{\mathbf{e}_1} + \mathbf{e}_2\widehat{Y}_{\mathbf{e}_2}$ if and only if $\widehat{X}_{\mathbf{e}_1} = \widehat{Y}_{\mathbf{e}_1}$ and $\widehat{X}_{\mathbf{e}_2} = \widehat{Y}_{\mathbf{e}_2}$.*

Proof. If $\mathbf{e}_1\widehat{X}_{\mathbf{e}_1} + \mathbf{e}_2\widehat{X}_{\mathbf{e}_2} = \mathbf{e}_1\widehat{Y}_{\mathbf{e}_1} + \mathbf{e}_2\widehat{Y}_{\mathbf{e}_2}$, then we have $\mathbf{e}_1(\widehat{X}_{\mathbf{e}_1} - \widehat{Y}_{\mathbf{e}_1}) + \mathbf{e}_2(\widehat{X}_{\mathbf{e}_2} - \widehat{Y}_{\mathbf{e}_2}) = \widehat{0}$. Suppose now that $\{\widehat{m}_l \mid l \in \{1, \dots, n\}\}$ is a free basis of M , then we have $\widehat{X}_{\mathbf{e}_k} = \sum_{l=1}^n x_{kl}\widehat{m}_l$ and $\widehat{Y}_{\mathbf{e}_k} = \sum_{l=1}^n y_{kl}\widehat{m}_l$ ($k = 1, 2$), $x_{kl}, y_{kl} \in \mathbb{C}(\mathbf{i}_1)$. Therefore, we find

$$\begin{aligned} \widehat{0} &= \mathbf{e}_1(\widehat{X}_{\mathbf{e}_1} - \widehat{Y}_{\mathbf{e}_1}) + \mathbf{e}_2(\widehat{X}_{\mathbf{e}_2} - \widehat{Y}_{\mathbf{e}_2}) \\ &= \mathbf{e}_1 \left(\sum_{l=1}^n x_{1l}\widehat{m}_l - \sum_{l=1}^n y_{1l}\widehat{m}_l \right) + \mathbf{e}_2 \left(\sum_{l=1}^n x_{2l}\widehat{m}_l - \sum_{l=1}^n y_{2l}\widehat{m}_l \right) \\ &= \sum_{l=1}^n (x_l - y_l)\widehat{m}_l, \end{aligned}$$

where $x_l := \mathbf{e}_1x_{1l} + \mathbf{e}_2x_{2l} \in \mathbb{T}$ and $y_l := \mathbf{e}_1y_{1l} + \mathbf{e}_2y_{2l} \in \mathbb{T}$. This implies that $x_l = y_l$ for all $l \in \{1, \dots, n\}$; in other words $x_{1l} = y_{2l}$ and $x_{2l} = y_{1l}$, i.e. $\widehat{X}_{\mathbf{e}_k} = \widehat{Y}_{\mathbf{e}_k}$ for $k = 1, 2$.

Conversely, if $\widehat{X}_{\mathbf{e}_1} = \widehat{Y}_{\mathbf{e}_1}$ and $\widehat{X}_{\mathbf{e}_2} = \widehat{Y}_{\mathbf{e}_2}$ we find trivially the desired result.

Whenever $\widehat{X} \in M$, we define the projection $P_k : M \rightarrow V$ as

$$P_k(\widehat{X}) := \widehat{X}_{\mathbf{e}_k} \quad (3.2)$$

for $k = 1, 2$. This definition is a generalization of the mutually complementary projections $\{P_1, P_2\}$ defined in [11] on \mathbb{T} , where \mathbb{T} is considered as the canonical \mathbb{T} -module over the ring of bicomplex numbers. Moreover, from Corollary 1, $\widehat{X}_{\mathbf{e}_1}$ and $\widehat{X}_{\mathbf{e}_2}$ are uniquely determined from a given \mathbb{T} -basis and the projections P_1 and P_2 satisfies the following property:

$$P_k(w_1\widehat{X} + w_2\widehat{Y}) = P_k(w_1)P_k(\widehat{X}) + P_k(w_2)P_k(\widehat{Y}) \quad (3.3)$$

$\forall w_1, w_2 \in \mathbb{T}, \forall \widehat{X}, \widehat{Y} \in M$ and $k = 1, 2$.

The vector space V is defined from the free \mathbb{T} -module M with a given \mathbb{T} -basis. The next theorem tell us that M is isomorphic to $V^2 = \{(\widehat{X}; \widehat{Y}) \mid \widehat{X}, \widehat{Y} \in V\}$, where the addition $+_{V^2}$ and the multiplication \cdot_{V^2} by a scalar are defined by

$$\begin{aligned} +_{V^2} : \quad & V^2 \times V^2 \rightarrow V^2 \\ & ((\widehat{X}_1; \widehat{Y}_1), (\widehat{X}_2; \widehat{Y}_2)) \mapsto (\widehat{X}_1; \widehat{Y}_1) +_{V^2} (\widehat{X}_2; \widehat{Y}_2) \\ & \quad \quad \quad := (\widehat{X}_1 + \widehat{X}_2; \widehat{Y}_1 + \widehat{Y}_2) \\ \cdot_{V^2} : \quad & \mathbb{T} \times V^2 \rightarrow V^2 \\ & (\lambda, (\widehat{X}; \widehat{Y})) \mapsto \lambda \cdot_{V^2} (\widehat{X}; \widehat{Y}) \\ & \quad \quad \quad := (\lambda_1\widehat{X}; \lambda_2\widehat{Y}), \end{aligned}$$

where $\lambda = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2$. Here the symbol $+$ denotes the addition on V and $\lambda_1 \widehat{X}$ or $\lambda_2 \widehat{Y}$ denotes the multiplication by a scalar on V (which are also the addition and the multiplication defined on M). Note that we use the notation $(\widehat{X}; \widehat{Y})$ to denote an element of V^2 , instead of the usual notation $(\widehat{X}, \widehat{Y})$, to avoid confusion with the bicomplex scalar product defined below.

Theorem 2. *The set V^2 defined with the addition $+_{V^2}$ and the multiplication by a scalar \cdot_{V^2} over the bicomplex numbers \mathbb{T} is isomorphic to M , i.e.*

$$(V^2, +_{V^2}, \cdot_{V^2}) \simeq (M, +, \cdot).$$

Proof. First, it is easy to show that V^2 is a \mathbb{T} -module with $+_{V^2}$ and \cdot_{V^2} defined above. Now let us consider the function $\Phi : V^2 \rightarrow M$ defined by $\Phi((\widehat{X}; \widehat{Y})) = \mathbf{e}_1 \widehat{X} + \mathbf{e}_2 \widehat{Y}$. It is not difficult to show that $\Phi((\widehat{X}_1; \widehat{Y}_1) +_{V^2} (\widehat{X}_2; \widehat{Y}_2)) = \Phi((\widehat{X}_1; \widehat{Y}_1)) + \Phi((\widehat{X}_2; \widehat{Y}_2))$ and that $\Phi(\lambda \cdot_{V^2} \widehat{X}) = \lambda \Phi(\widehat{X})$, i.e. that Φ is an homomorphism. The function Φ is a one-to-one function. Indeed if $\Phi((\widehat{X}_1; \widehat{Y}_1)) = \Phi((\widehat{X}_2; \widehat{Y}_2))$, then $\mathbf{e}_1 \widehat{X}_1 + \mathbf{e}_2 \widehat{Y}_1 = \mathbf{e}_1 \widehat{X}_2 + \mathbf{e}_2 \widehat{Y}_2$ which implies that $\widehat{X}_1 = \widehat{X}_2$ and $\widehat{Y}_1 = \widehat{Y}_2$ from Corollary 1. Finally, Φ is an onto function since for all $\widehat{X} = \mathbf{e}_1 \widehat{X}_{\mathbf{e}_1} + \mathbf{e}_2 \widehat{X}_{\mathbf{e}_2} \in M$, we have $\Phi((\widehat{X}_{\mathbf{e}_1}; \widehat{X}_{\mathbf{e}_2})) = \widehat{X}$.

Theorem 3. *Let $\{\widehat{v}_l \mid l \in \{1, \dots, n\}\}$ be a basis of the vector space V over $\mathbb{C}(\mathbf{i}_1)$.*

Then $\{(\widehat{v}_l; \widehat{v}_l) \mid l \in \{1, \dots, n\}\}$ is a basis of the free \mathbb{T} -module $(V^2, +_{V^2}, \cdot_{V^2})$ and $\{\widehat{v}_l \mid l \in \{1, \dots, n\}\}$ is a \mathbb{T} -basis of M .

Proof. Let us consider an arbitrary $(\widehat{X}; \widehat{Y}) \in V^2$, then

$$(\widehat{X}; \widehat{Y}) = \left(\sum_{l=1}^n c_{1l} \widehat{v}_l; \sum_{l=1}^n c_{2l} \widehat{v}_l \right) = \sum_{l=1}^n (c_{1l} \widehat{v}_l; c_{2l} \widehat{v}_l),$$

with $c_{kl} \in \mathbb{C}(\mathbf{i}_1)$ ($k = 1, 2$). Here the summations in the second expression are the addition on V and the summation in the third expression is the addition over V^2 , i.e. the addition $+_{V^2}$. Therefore, we have

$$(\widehat{X}; \widehat{Y}) = \sum_{l=1}^n c_l \cdot_{V^2} (\widehat{v}_l; \widehat{v}_l),$$

where $c_l = \mathbf{e}_1 c_{1l} + \mathbf{e}_2 c_{2l} \in \mathbb{T}$. Moreover, if $(\widehat{X}; \widehat{Y}) = (\widehat{0}; \widehat{0})$, then $c_{1l} = c_{2l} = 0$ for all $l \in \{1, \dots, n\}$ since $\{\widehat{v}_l \mid l \in \{1, \dots, n\}\}$ is a basis of V and $c_l = 0$ for all $l \in \{1, \dots, n\}$. Therefore $\{(\widehat{v}_l; \widehat{v}_l) \mid l \in \{1, \dots, n\}\}$ is a \mathbb{T} -basis of V^2 and the \mathbb{T} -module $(V^2, +_{V^2}, \cdot_{V^2})$ is free. It is now easy to see that $\{\widehat{v}_l \mid l \in \{1, \dots, n\}\}$ is a \mathbb{T} -basis of M since the isomorphism Φ given in the proof of Theorem 2 gives $\Phi((\widehat{v}_l; \widehat{v}_l)) = \mathbf{e}_1 \widehat{v}_l + \mathbf{e}_2 \widehat{v}_l = \widehat{v}_l$ for all $l \in \{1, \dots, n\}$.

Remark. For $(\widehat{X}; \widehat{Y}) \in V^2$, we have

$$\begin{aligned} (\widehat{X}; \widehat{Y}) &= (\widehat{X}; \widehat{0}) +_{V^2} (\widehat{0}; \widehat{Y}) \\ &= (\mathbf{1e}_1 + 0\mathbf{e}_2) \cdot_{V^2} (\widehat{X}; \widehat{X}) +_{V^2} (0\mathbf{e}_1 + \mathbf{1e}_2) \cdot_{V^2} (\widehat{Y}; \widehat{Y}) \\ &= \mathbf{e}_1 \cdot_{V^2} (\widehat{X}; \widehat{X}) +_{V^2} \mathbf{e}_2 \cdot_{V^2} (\widehat{Y}; \widehat{Y}), \end{aligned}$$

where $(\widehat{X}; \widehat{X})$ and $(\widehat{Y}; \widehat{Y})$ are in the vector space $V' := \left\{ \sum_{l=1}^n c_l \cdot_{V^2} (\widehat{v}_l; \widehat{v}_l) \mid c_l \in \mathbb{C}(\mathbf{i}_1) \right\}$ associated with the free \mathbb{T} -module V^2 using the \mathbb{T} -basis $\left\{ (\widehat{v}_l; \widehat{v}_l) \mid l \in \{1, \dots, n\} \right\}$.

Now, from Theorem 3 we obtain the following corollary.

Corollary 2. *Let M be a free \mathbb{T} -module with a finite \mathbb{T} -basis. The submodule vector space V associated with M is invariant under a new \mathbb{T} -basis of M generated by another basis of V .*

4. Bicomplex Scalar Product

Let us begin with a preliminary definition.

Definition 1. A hyperbolic number $w = a\mathbf{e}_1 + b\mathbf{e}_2$ is defined to be positive if $a, b \in \mathbb{R}^+$. We denote the set of all positive hyperbolic numbers by

$$\mathbb{D}^+ := \{a\mathbf{e}_1 + b\mathbf{e}_2 \mid a, b \geq 0\}.$$

We are now able to give a definition of a bicomplex scalar product. (In this article, the physicist convention will be used for the order of the elements in the bicomplex scalar product.)

Definition 2. Let M be a free \mathbb{T} -module of finite dimension. With each pair \widehat{X} and \widehat{Y} in M , taken in this order, we associate a bicomplex number, which is their bicomplex scalar product $(\widehat{X}, \widehat{Y})$, and which satisfies the following properties:

1. $(\widehat{X}, \widehat{Y}_1 + \widehat{Y}_2) = (\widehat{X}, \widehat{Y}_1) + (\widehat{X}, \widehat{Y}_2)$, $\forall \widehat{X}, \widehat{Y}_1, \widehat{Y}_2 \in M$;
2. $(\widehat{X}, \alpha\widehat{Y}) = \alpha(\widehat{X}, \widehat{Y})$, $\forall \alpha \in \mathbb{T}$, $\forall \widehat{X}, \widehat{Y} \in M$;
3. $(\widehat{X}, \widehat{Y}) = (\widehat{Y}, \widehat{X})^{\dagger 3}$, $\forall \widehat{X}, \widehat{Y} \in M$;
4. $(\widehat{X}, \widehat{X}) = 0 \Leftrightarrow \widehat{X} = 0$, $\forall \widehat{X} \in M$.

As a consequence of property 3, we have that $(\widehat{X}, \widehat{X}) \in \mathbb{D}$. Note that definition 2 is a general definition of a bicomplex scalar product. However, in this article we will also require the bicomplex scalar product (\cdot, \cdot) to be *hyperbolic positive*, i.e.

$$(\widehat{X}, \widehat{X}) \in \mathbb{D}^+, \forall \widehat{X} \in M \quad (4.1)$$

and *closed* under the vector space V , i.e.

$$(\widehat{X}, \widehat{Y}) \in \mathbb{C}(\mathbf{i}_1), \quad \forall \widehat{X}, \widehat{Y} \in V. \quad (4.2)$$

For the rest of this paper, we will assume a given \mathbb{T} -basis for M , which implies a given vector space V .

Theorem 4. *Let $\widehat{X}, \widehat{Y} \in M$, then*

$$(\widehat{X}, \widehat{Y}) = \mathbf{e}_1(\widehat{X}_{\mathbf{e}_1}, \widehat{Y}_{\mathbf{e}_1}) + \mathbf{e}_2(\widehat{X}_{\mathbf{e}_2}, \widehat{Y}_{\mathbf{e}_2}) \quad (4.3)$$

and

$$P_k((\widehat{X}, \widehat{Y})) = (\widehat{X}, \widehat{Y})_{\mathbf{e}_k} = (\widehat{X}_{\mathbf{e}_k}, \widehat{Y}_{\mathbf{e}_k}) \in \mathbb{C}(\mathbf{i}_1) \quad (4.4)$$

for $k = 1, 2$.

Proof. From equation (3.2), it comes automatically that $P_k((\widehat{X}, \widehat{Y})) = (\widehat{X}, \widehat{Y})_{\mathbf{e}_k} \in \mathbb{C}(\mathbf{i}_1)$ for $k = 1, 2$. Let $\widehat{X} = \mathbf{e}_1\widehat{X}_{\mathbf{e}_1} + \mathbf{e}_2\widehat{X}_{\mathbf{e}_2}$ and $\widehat{Y} = \mathbf{e}_1\widehat{Y}_{\mathbf{e}_1} + \mathbf{e}_2\widehat{Y}_{\mathbf{e}_2}$, then using the properties of the bicomplex scalar product, we also have

$$\begin{aligned} (\widehat{X}, \widehat{Y}) &= (\mathbf{e}_1\widehat{X}_{\mathbf{e}_1} + \mathbf{e}_2\widehat{X}_{\mathbf{e}_2}, \mathbf{e}_1\widehat{Y}_{\mathbf{e}_1} + \mathbf{e}_2\widehat{Y}_{\mathbf{e}_2}) \\ &= (\mathbf{e}_1\widehat{X}_{\mathbf{e}_1} + \mathbf{e}_2\widehat{X}_{\mathbf{e}_2}, \mathbf{e}_1\widehat{Y}_{\mathbf{e}_1}) + (\mathbf{e}_1\widehat{X}_{\mathbf{e}_1} + \mathbf{e}_2\widehat{X}_{\mathbf{e}_2}, \mathbf{e}_2\widehat{Y}_{\mathbf{e}_2}) \\ &= (\mathbf{e}_1\widehat{Y}_{\mathbf{e}_1}, \mathbf{e}_1\widehat{X}_{\mathbf{e}_1} + \mathbf{e}_2\widehat{X}_{\mathbf{e}_2})^{\dagger 3} + (\mathbf{e}_2\widehat{Y}_{\mathbf{e}_2}, \mathbf{e}_1\widehat{X}_{\mathbf{e}_1} + \mathbf{e}_2\widehat{X}_{\mathbf{e}_2})^{\dagger 3} \\ &= (\mathbf{e}_1\widehat{Y}_{\mathbf{e}_1}, \mathbf{e}_1\widehat{X}_{\mathbf{e}_1})^{\dagger 3} + (\mathbf{e}_1\widehat{Y}_{\mathbf{e}_1}, \mathbf{e}_2\widehat{X}_{\mathbf{e}_2})^{\dagger 3} \\ &\quad + (\mathbf{e}_2\widehat{Y}_{\mathbf{e}_2}, \mathbf{e}_1\widehat{X}_{\mathbf{e}_1})^{\dagger 3} + (\mathbf{e}_2\widehat{Y}_{\mathbf{e}_2}, \mathbf{e}_2\widehat{X}_{\mathbf{e}_2})^{\dagger 3} \\ &= \mathbf{e}_1^{\dagger 3}(\mathbf{e}_1\widehat{Y}_{\mathbf{e}_1}, \widehat{X}_{\mathbf{e}_1})^{\dagger 3} + \mathbf{e}_2^{\dagger 3}(\mathbf{e}_1\widehat{Y}_{\mathbf{e}_1}, \widehat{X}_{\mathbf{e}_2})^{\dagger 3} \\ &\quad + \mathbf{e}_1^{\dagger 3}(\mathbf{e}_2\widehat{Y}_{\mathbf{e}_2}, \widehat{X}_{\mathbf{e}_1})^{\dagger 3} + \mathbf{e}_2^{\dagger 3}(\mathbf{e}_2\widehat{Y}_{\mathbf{e}_2}, \widehat{X}_{\mathbf{e}_2})^{\dagger 3} \\ &= \mathbf{e}_1^{\dagger 3}\mathbf{e}_1(\widehat{X}_{\mathbf{e}_1}, \widehat{Y}_{\mathbf{e}_1}) + \mathbf{e}_2^{\dagger 3}\mathbf{e}_1(\widehat{X}_{\mathbf{e}_2}, \widehat{Y}_{\mathbf{e}_1}) \\ &\quad + \mathbf{e}_1^{\dagger 3}\mathbf{e}_2(\widehat{X}_{\mathbf{e}_1}, \widehat{Y}_{\mathbf{e}_2}) + \mathbf{e}_2^{\dagger 3}\mathbf{e}_2(\widehat{X}_{\mathbf{e}_2}, \widehat{Y}_{\mathbf{e}_2}) \\ &= \mathbf{e}_1(\widehat{X}_{\mathbf{e}_1}, \widehat{Y}_{\mathbf{e}_1}) + \mathbf{e}_2(\widehat{X}_{\mathbf{e}_2}, \widehat{Y}_{\mathbf{e}_2}). \end{aligned}$$

Hence,

$$(\widehat{X}, \widehat{Y}) = \mathbf{e}_1(\widehat{X}_{\mathbf{e}_1}, \widehat{Y}_{\mathbf{e}_1}) + \mathbf{e}_2(\widehat{X}_{\mathbf{e}_2}, \widehat{Y}_{\mathbf{e}_2})$$

and, from property (4.2), we obtain

$$P_k((\widehat{X}, \widehat{Y})) = (\widehat{X}, \widehat{Y})_{\mathbf{e}_k} = (\widehat{X}_{\mathbf{e}_k}, \widehat{Y}_{\mathbf{e}_k}) \in \mathbb{C}(\mathbf{i}_1)$$

for $k = 1, 2$.

Theorem 5. $\{V; (\cdot, \cdot)\}$ is a complex (in $\mathbb{C}(\mathbf{i}_1)$) pre-Hilbert space.

Proof. By definition, $V \subseteq M$. Hence, we obtain automatically that:

1. $(\widehat{X}, \widehat{Y}_1 + \widehat{Y}_2) = (\widehat{X}, \widehat{Y}_1) + (\widehat{X}, \widehat{Y}_2), \forall \widehat{X}, \widehat{Y}_1, \widehat{Y}_2 \in V;$
2. $(\widehat{X}, \alpha\widehat{Y}) = \alpha(\widehat{X}, \widehat{Y}), \forall \alpha \in \mathbb{C}(\mathbf{i}_1)$ and $\forall \widehat{X}, \widehat{Y} \in V;$
3. $(\widehat{X}, \widehat{X}) = 0 \Leftrightarrow \widehat{X} = 0, \forall \widehat{X} \in V.$

Moreover, the fact that $(\widehat{X}, \widehat{Y}) \in \mathbb{C}(\mathbf{i}_1)$ implies that $(\widehat{X}, \widehat{Y}) = (\widehat{Y}, \widehat{X})^{\dagger_3} = \overline{(\widehat{Y}, \widehat{X})}$ and $(\widehat{X}, \widehat{X}) \in \mathbb{D}^+ \cap \mathbb{C}(\mathbf{i}_1) = \mathbb{R}^+$. Hence, $\{V; (\cdot, \cdot)\}$ is a complex (in $\mathbb{C}(\mathbf{i}_1)$) pre-Hilbert space.

Remark. We note that the results obtained in this theorem are still valid by using \dagger_1 instead of \dagger_3 in the definition of the bicomplex scalar product.

Let us denote $\|\widehat{X}\| := (\widehat{X}, \widehat{X})^{\frac{1}{2}}, \forall \widehat{X} \in V$.

Corollary 3. *Let $\widehat{X} \in V$. The function $\widehat{X} \mapsto \|\widehat{X}\| \geq 0$ is a norm on V .*

Corollary 4. *Let $\widehat{X} \in M$; then*

$$P_k((\widehat{X}, \widehat{X})) = (\widehat{X}, \widehat{X})_{\mathbf{e}_k} = (\widehat{X}_{\mathbf{e}_k}, \widehat{X}_{\mathbf{e}_k}) = \|\widehat{X}_{\mathbf{e}_k}\|^2$$

for $k = 1, 2$.

Now, let us extend this norm on M with the following function:

$$\|\widehat{X}\| := \left| (\widehat{X}, \widehat{X})^{\frac{1}{2}} \right| = \left| \mathbf{e}_1 \|\widehat{X}_{\mathbf{e}_1}\| + \mathbf{e}_2 \|\widehat{X}_{\mathbf{e}_2}\| \right|, \forall \widehat{X} \in M. \quad (4.5)$$

This *norm* has the following properties.

Theorem 6. *Let $\widehat{X}, \widehat{Y} \in M$ and $d(\widehat{X}, \widehat{Y}) := \|\widehat{X} - \widehat{Y}\|$; then*

1. $\|\widehat{X}\| \geq 0$
2. $\|\widehat{X}\| = 0 \Leftrightarrow \widehat{X} = 0$
3. $\|\alpha\widehat{X}\| = |\alpha| \|\widehat{X}\|, \forall \alpha \in \mathbb{C}(\mathbf{i}_1)$ or $\mathbb{C}(\mathbf{i}_2)$
4. $\|\alpha\widehat{X}\| \leq \sqrt{2} |\alpha|_3 \|\widehat{X}\|, \forall \alpha \in \mathbb{T}$
5. $\|\widehat{X} + \widehat{Y}\| \leq \|\widehat{X}\| + \|\widehat{Y}\|$
6. $\{M, d\}$ is a **metric space**.

Proof. The proof of 1 and 2 comes directly from equation (4.5). Let $\widehat{X} = \mathbf{e}_1 \widehat{X}_{\mathbf{e}_1} + \mathbf{e}_2 \widehat{X}_{\mathbf{e}_2} \in M$ and $\alpha \in \mathbb{C}(\mathbf{i}_1)$ or $\mathbb{C}(\mathbf{i}_2)$, then

$$\begin{aligned} \|\alpha\widehat{X}\| &= \left| (\alpha\widehat{X}, \alpha\widehat{X})^{\frac{1}{2}} \right| \\ &= \left| (\alpha\bar{\alpha}(\widehat{X}, \widehat{X}))^{\frac{1}{2}} \right| \\ &= \left| \left(\mathbf{e}_1 |\alpha|^2 (\widehat{X}, \widehat{X})_{\mathbf{e}_1} + \mathbf{e}_2 |\alpha|^2 (\widehat{X}, \widehat{X})_{\mathbf{e}_2} \right)^{\frac{1}{2}} \right| \\ &= \left| \mathbf{e}_1 |\alpha| (\widehat{X}, \widehat{X})_{\mathbf{e}_1}^{\frac{1}{2}} + \mathbf{e}_2 |\alpha| (\widehat{X}, \widehat{X})_{\mathbf{e}_2}^{\frac{1}{2}} \right| \\ &= |\alpha| \left| \mathbf{e}_1 \|\widehat{X}_{\mathbf{e}_1}\| + \mathbf{e}_2 \|\widehat{X}_{\mathbf{e}_2}\| \right| \\ &= |\alpha| \|\widehat{X}\|. \end{aligned}$$

More generally, if $\alpha \in \mathbb{T}$, we obtain

$$\begin{aligned}
\| \alpha \widehat{X} \| &= |(\alpha \widehat{X}, \alpha \widehat{X})^{\frac{1}{2}}| \\
&= |(\alpha \alpha^{\dagger 3} (\widehat{X}, \widehat{X}))^{\frac{1}{2}}| \\
&= |(|\alpha|_{\mathbf{j}}^2 (\widehat{X}, \widehat{X}))^{\frac{1}{2}}| \\
&= |\alpha|_{\mathbf{j}} (\widehat{X}, \widehat{X})^{\frac{1}{2}}| \\
&= |\alpha|_{\mathbf{j}} \| \widehat{X} \| \\
&\leq \sqrt{2} |\alpha|_{\mathbf{j}} \| \widehat{X} \| \\
&= \sqrt{2} |\alpha|_{\mathbf{3}} \| \widehat{X} \|.
\end{aligned}$$

To complete the proof, we need to establish a triangular inequality over the \mathbb{T} -module M . Let $\widehat{X}, \widehat{Y} \in M$, then

$$\begin{aligned}
\| \widehat{X} + \widehat{Y} \| &= |(\widehat{X} + \widehat{Y}, \widehat{X} + \widehat{Y})^{\frac{1}{2}}| \\
&= |e_1 \| (\widehat{X} + \widehat{Y})_{e_1} \| + e_2 \| (\widehat{X} + \widehat{Y})_{e_2} \| | \\
&= |e_1 \| \widehat{X}_{e_1} + \widehat{Y}_{e_1} \| + e_2 \| \widehat{X}_{e_2} + \widehat{Y}_{e_2} \| | \\
&= \left(\frac{\| \widehat{X}_{e_1} + \widehat{Y}_{e_1} \|^2 + \| \widehat{X}_{e_2} + \widehat{Y}_{e_2} \|^2}{2} \right)^{\frac{1}{2}} \\
&\leq \left(\frac{(\| \widehat{X}_{e_1} \| + \| \widehat{Y}_{e_1} \|^2 + (\| \widehat{X}_{e_2} \| + \| \widehat{Y}_{e_2} \|^2)}{2} \right)^{\frac{1}{2}} \\
&= |e_1 (\| \widehat{X}_{e_1} \| + \| \widehat{Y}_{e_1} \|) + e_2 (\| \widehat{X}_{e_2} \| + \| \widehat{Y}_{e_2} \|) | \\
&= |(e_1 \| \widehat{X}_{e_1} \| + e_2 \| \widehat{X}_{e_2} \|) + (e_1 \| \widehat{Y}_{e_1} \| + e_2 \| \widehat{Y}_{e_2} \|) | \\
&\leq \| \widehat{X} \| + \| \widehat{Y} \|.
\end{aligned}$$

Now, using properties 1, 2, 3 and 5, it is easy to obtain that $\{M, d\}$ is a metric space.

With the bicomplex scalar product, it is possible to obtain a bicomplex version of the well known Schwarz inequality.

Theorem 7 (Bicomplex Schwarz inequality). *Let $\widehat{X}, \widehat{Y} \in M$; then*

$$|(\widehat{X}, \widehat{Y})| \leq |(\widehat{X}, \widehat{X})^{\frac{1}{2}} (\widehat{Y}, \widehat{Y})^{\frac{1}{2}}| \leq \sqrt{2} \| \widehat{X} \| \| \widehat{Y} \|.$$

Proof. From the complex (in $\mathbb{C}(\mathbf{i}_1)$) Schwarz inequality we have that

$$|(\widehat{X}, \widehat{Y})| \leq \| \widehat{X} \| \| \widehat{Y} \| \quad \forall \widehat{X}, \widehat{Y} \in V. \quad (4.6)$$

Therefore, if $\widehat{X}, \widehat{Y} \in M$, we obtain

$$\begin{aligned}
|(\widehat{X}, \widehat{Y})| &= |\mathbf{e}_1(\widehat{X}, \widehat{Y})_{\mathbf{e}_1} + \mathbf{e}_2(\widehat{X}, \widehat{Y})_{\mathbf{e}_2}| \\
&= |\mathbf{e}_1(\widehat{X}_{\mathbf{e}_1}, \widehat{Y}_{\mathbf{e}_1}) + \mathbf{e}_2(\widehat{X}_{\mathbf{e}_2}, \widehat{Y}_{\mathbf{e}_2})| \\
&= \left(\frac{|(\widehat{X}_{\mathbf{e}_1}, \widehat{Y}_{\mathbf{e}_1})|^2 + |(\widehat{X}_{\mathbf{e}_2}, \widehat{Y}_{\mathbf{e}_2})|^2}{2} \right)^{\frac{1}{2}} \\
&\leq \left(\frac{\|\widehat{X}_{\mathbf{e}_1}\|^2 \|\widehat{Y}_{\mathbf{e}_1}\|^2 + \|\widehat{X}_{\mathbf{e}_2}\|^2 \|\widehat{Y}_{\mathbf{e}_2}\|^2}{2} \right)^{\frac{1}{2}} \\
&= |\mathbf{e}_1| \|\widehat{X}_{\mathbf{e}_1}\| \|\widehat{Y}_{\mathbf{e}_1}\| + |\mathbf{e}_2| \|\widehat{X}_{\mathbf{e}_2}\| \|\widehat{Y}_{\mathbf{e}_2}\| \\
&= |(\widehat{X}, \widehat{X})|^{\frac{1}{2}} |(\widehat{Y}, \widehat{Y})|^{\frac{1}{2}}.
\end{aligned}$$

Hence, $|(\widehat{X}, \widehat{Y})| \leq |(\widehat{X}, \widehat{X})|^{\frac{1}{2}} |(\widehat{Y}, \widehat{Y})|^{\frac{1}{2}} \leq \sqrt{2} \|\widehat{X}\| \|\widehat{Y}\|$.

5. Hyperbolic Scalar Product

From the preceding section, it is now easy to define the hyperbolic version of the bicomplex scalar product.

Definition 3. Let M be a free \mathbb{D} -module of finite dimension. With each pair \widehat{X} and \widehat{Y} in M , taken in this order, we associate a hyperbolic number, which is their hyperbolic scalar product $(\widehat{X}, \widehat{Y})$, and which satisfies the following properties:

1. $(\widehat{X}, \widehat{Y}_1 + \widehat{Y}_2) = (\widehat{X}, \widehat{Y}_1) + (\widehat{X}, \widehat{Y}_2)$
2. $(\widehat{X}, \alpha \widehat{Y}) = \alpha (\widehat{X}, \widehat{Y}), \forall \alpha \in \mathbb{D}$
3. $(\widehat{X}, \widehat{Y}) = (\widehat{Y}, \widehat{X})$
4. $(\widehat{X}, \widehat{X}) = 0 \Leftrightarrow \widehat{X} = 0$.

All definitions and results of Section 4 can be applied directly in the hyperbolic case if the hyperbolic scalar product (\cdot, \cdot) is *hyperbolic positive* i.e.

$$(\widehat{X}, \widehat{X}) \in \mathbb{D}^+ \quad \forall \widehat{X} \in M \quad (5.1)$$

and *closed* under the real vector space $V := \left\{ \sum_{l=1}^n x_l \widehat{m}_l \mid x_l \in \mathbb{R} \right\}$ i.e.

$$(\widehat{X}, \widehat{Y}) \in \mathbb{C}(\mathbf{i}_1) \cap \mathbb{D} = \mathbb{R} \quad \forall \widehat{X}, \widehat{Y} \in V \quad (5.2)$$

for a specific \mathbb{D} -basis $\{\widehat{m}_l \mid l \in \{1, \dots, n\}\}$ of M . In particular, we obtain a hyperbolic Schwarz inequality. Moreover, it is always possible to obtain the angle θ ,

between \widehat{X} and \widehat{Y} in V , with the following well known formula:

$$\cos \theta = \frac{(\widehat{X}, \widehat{Y})}{\|\widehat{X}\| \|\widehat{Y}\|}. \quad (5.3)$$

From this result, we can derive the following analogous result for the \mathbb{D} -module M .

Theorem 8. Let $\widehat{X}, \widehat{Y} \in M$ and θ_k be the angle between $\widehat{X}_{\mathbf{e}_k}$ and $\widehat{Y}_{\mathbf{e}_k}$ for $k = 1, 2$. Then,

$$\cos \left(\frac{\theta_1 + \theta_2}{2} + \frac{\theta_1 - \theta_2}{2} \mathbf{j} \right) = \frac{(\widehat{X}, \widehat{Y})}{(\widehat{X}, \widehat{X})^{\frac{1}{2}} (\widehat{Y}, \widehat{Y})^{\frac{1}{2}}}.$$

Proof. From the identity (5.3), we have

$$\begin{aligned} (\cos \theta_1) \mathbf{e}_1 + (\cos \theta_2) \mathbf{e}_2 &= \frac{(\widehat{X}_{\mathbf{e}_1}, \widehat{Y}_{\mathbf{e}_1})}{\|\widehat{X}_{\mathbf{e}_1}\| \|\widehat{Y}_{\mathbf{e}_1}\|} \mathbf{e}_1 + \frac{(\widehat{X}_{\mathbf{e}_2}, \widehat{Y}_{\mathbf{e}_2})}{\|\widehat{X}_{\mathbf{e}_2}\| \|\widehat{Y}_{\mathbf{e}_2}\|} \mathbf{e}_2 \\ &= \frac{(\widehat{X}, \widehat{Y})}{(\widehat{X}, \widehat{X})^{\frac{1}{2}} (\widehat{Y}, \widehat{Y})^{\frac{1}{2}}}. \end{aligned}$$

Moreover, it is easy to show that $\cos(\theta_1 \mathbf{e}_1 + \theta_2 \mathbf{e}_2) = (\cos \theta_1) \mathbf{e}_1 + (\cos \theta_2) \mathbf{e}_2$ and $\theta_1 \mathbf{e}_1 + \theta_2 \mathbf{e}_2 = \frac{\theta_1 + \theta_2}{2} + \frac{\theta_1 - \theta_2}{2} \mathbf{j}$ (see [12]). Hence,

$$\cos \left(\frac{\theta_1 + \theta_2}{2} + \frac{\theta_1 - \theta_2}{2} \mathbf{j} \right) = \frac{(\widehat{X}, \widehat{Y})}{(\widehat{X}, \widehat{X})^{\frac{1}{2}} (\widehat{Y}, \widehat{Y})^{\frac{1}{2}}}.$$

From this result, it is now possible to define the ‘‘hyperbolic angle’’ between two elements of a \mathbb{D} -module M .

Definition 4. Let $\widehat{X}, \widehat{Y} \in M$ and θ_k be the angle between $\widehat{X}_{\mathbf{e}_k}$ and $\widehat{Y}_{\mathbf{e}_k}$ for $k = 1, 2$. We define the hyperbolic angle between \widehat{X} and \widehat{Y} as

$$\frac{\theta_1 + \theta_2}{2} + \frac{\theta_1 - \theta_2}{2} \mathbf{j}.$$

We note that our definition of the hyperbolic scalar product is different from the definitions given in [8, 9] and [19].

6. Bicomplex Hilbert Space

Definition 5. Let M be a free \mathbb{T} -module with a finite \mathbb{T} -basis. Let also (\cdot, \cdot) be a bicomplex scalar product defined on M . The space $\{M, (\cdot, \cdot)\}$ is called a \mathbb{T} -inner product space.

Definition 6. A complete \mathbb{T} -inner product space is called a \mathbb{T} -Hilbert space.

Lemma 1. Let $\widehat{X} \in M$; then

$$\|\widehat{X}_{\mathbf{e}_k}\| \leq \sqrt{2} \|\widehat{X}\|, \text{ for } k = 1, 2.$$

Proof. For $k = 1, 2$, we have

$$\begin{aligned} \|\widehat{X}_{\mathbf{e}_k}\| &\leq \sqrt{2} \left(\frac{\|\widehat{X}_{\mathbf{e}_1}\|^2 + \|\widehat{X}_{\mathbf{e}_2}\|^2}{2} \right)^{\frac{1}{2}} \\ &= \sqrt{2} |\mathbf{e}_1| \|\widehat{X}_{\mathbf{e}_1}\| + |\mathbf{e}_2| \|\widehat{X}_{\mathbf{e}_2}\| \\ &= \sqrt{2} \|\widehat{X}\|. \end{aligned}$$

Lemma 2. *The pre-Hilbert space $\{V, (\cdot, \cdot)\}$ is closed in the metric space $\{M, (\cdot, \cdot)\}$.*

Proof. Let $\widehat{X}_n = \mathbf{e}_1 \widehat{X}_n + \mathbf{e}_2 \widehat{X}_n \in V \forall n \in \mathbb{N}$ and $\widehat{X} = \mathbf{e}_1 \widehat{X}_{\mathbf{e}_1} + \mathbf{e}_2 \widehat{X}_{\mathbf{e}_2} \in M$. Suppose that $\widehat{X}_n \rightarrow \widehat{X}$ whenever $n \rightarrow \infty$ then $\|\widehat{X}_n - \widehat{X}\| \rightarrow 0$ as $n \rightarrow \infty$ i.e. $\|\widehat{X}_n - (\mathbf{e}_1 \widehat{X}_{\mathbf{e}_1} + \mathbf{e}_2 \widehat{X}_{\mathbf{e}_2})\| = \|(\mathbf{e}_1 \widehat{X}_n + \mathbf{e}_2 \widehat{X}_n) - (\mathbf{e}_1 \widehat{X}_{\mathbf{e}_1} + \mathbf{e}_2 \widehat{X}_{\mathbf{e}_2})\| = \|\mathbf{e}_1(\widehat{X}_n - \widehat{X}_{\mathbf{e}_1}) + \mathbf{e}_2(\widehat{X}_n - \widehat{X}_{\mathbf{e}_2})\| \rightarrow 0$ as $n \rightarrow \infty$. Therefore, from Lemma 1 we have that

$$\|\widehat{X}_n - \widehat{X}_{\mathbf{e}_k}\| \leq \sqrt{2} \|\mathbf{e}_1(\widehat{X}_n - \widehat{X}_{\mathbf{e}_1}) + \mathbf{e}_2(\widehat{X}_n - \widehat{X}_{\mathbf{e}_2})\| \rightarrow 0$$

as $n \rightarrow \infty$ for $k = 1, 2$. Hence, $\widehat{X}_{\mathbf{e}_1} = \widehat{X}_{\mathbf{e}_2} = \widehat{X}$ and $\widehat{X} = \mathbf{e}_1 \widehat{X} + \mathbf{e}_2 \widehat{X} \in V$.

Theorem 9. *A \mathbb{T} -inner product space $\{M, (\cdot, \cdot)\}$ is a \mathbb{T} -Hilbert space if and only if $\{V, (\cdot, \cdot)\}$ is a Hilbert space.*

Proof. From Theorem 5, $\{V, (\cdot, \cdot)\}$ is a pre-Hilbert space. So, we have to prove that $\{M, (\cdot, \cdot)\}$ is complete if and only if $\{V, (\cdot, \cdot)\}$ is complete. By definition $V \subseteq M$, therefore if M is complete then V is also complete since V is closed in M . Conversely, let $\widehat{X}_n = \mathbf{e}_1(\widehat{X}_n)_{\mathbf{e}_1} + \mathbf{e}_2(\widehat{X}_n)_{\mathbf{e}_2} \in M \forall n \in \mathbb{N}$ be a Cauchy sequence in M . Then, from Lemma 1, we have

$$\|(\widehat{X}_m)_{\mathbf{e}_k} - (\widehat{X}_n)_{\mathbf{e}_k}\| = \|(\widehat{X}_m - \widehat{X}_n)_{\mathbf{e}_k}\| \leq \sqrt{2} \|\widehat{X}_m - \widehat{X}_n\|$$

for $k = 1, 2$. So, $(\widehat{X}_n)_{\mathbf{e}_k}$ is also a Cauchy sequence in V for $k = 1, 2$. Therefore, there exist $\widehat{X}_{\mathbf{e}_1}, \widehat{X}_{\mathbf{e}_2} \in V$ such that $(\widehat{X}_n)_{\mathbf{e}_k} \rightarrow \widehat{X}_{\mathbf{e}_k}$ as $n \rightarrow \infty$ for $k = 1, 2$.

Now, from the triangular inequality, if we let $\widehat{X} := \mathbf{e}_1 \widehat{X}_{\mathbf{e}_1} + \mathbf{e}_2 \widehat{X}_{\mathbf{e}_2}$, then we obtain

$$\begin{aligned} \|\widehat{X}_n - \widehat{X}\| &= \|\mathbf{e}_1((\widehat{X}_n)_{\mathbf{e}_1} - \widehat{X}_{\mathbf{e}_1}) + \mathbf{e}_2((\widehat{X}_n)_{\mathbf{e}_2} - \widehat{X}_{\mathbf{e}_2})\| \\ &\leq \|\mathbf{e}_1((\widehat{X}_n)_{\mathbf{e}_1} - \widehat{X}_{\mathbf{e}_1})\| + \|\mathbf{e}_2((\widehat{X}_n)_{\mathbf{e}_2} - \widehat{X}_{\mathbf{e}_2})\| \\ &\leq \sqrt{2} |\mathbf{e}_1|_3 \|(\widehat{X}_n)_{\mathbf{e}_1} - \widehat{X}_{\mathbf{e}_1}\| \\ &\quad + \sqrt{2} |\mathbf{e}_2|_3 \|(\widehat{X}_n)_{\mathbf{e}_2} - \widehat{X}_{\mathbf{e}_2}\| \\ &= \|(\widehat{X}_n)_{\mathbf{e}_1} - \widehat{X}_{\mathbf{e}_1}\| + \|(\widehat{X}_n)_{\mathbf{e}_2} - \widehat{X}_{\mathbf{e}_2}\| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Hence, $\widehat{X}_n \rightarrow \widehat{X} \in M$ as $n \rightarrow \infty$.

Examples of Bicomplex Hilbert Spaces

1. Let us first consider $M = \mathbb{T}$, the canonical \mathbb{T} -module over the ring of bicomplex numbers. We consider now the trivial \mathbb{T} -basis $\{1\}$. In this case, the submodule vector space V is simply $V = \mathbb{C}(\mathbf{i}_1)$. Let $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_2$ be two scalar products on V . It is always possible to construct a general bicomplex scalar product as follows:

Let

$$w_1 = (z_{11} - z_{12}\mathbf{i}_1)\mathbf{e}_1 + (z_{11} + z_{12}\mathbf{i}_1)\mathbf{e}_2$$

and

$$w_2 = (z_{21} - z_{22}\mathbf{i}_1)\mathbf{e}_1 + (z_{21} + z_{22}\mathbf{i}_1)\mathbf{e}_2,$$

where, $z_{11}, z_{12}, z_{21}, z_{22} \in \mathbb{C}(\mathbf{i}_1)$. We define

$$(w_1, w_2) := (z_{11} - z_{12}\mathbf{i}_1, z_{21} - z_{22}\mathbf{i}_1)_1 \mathbf{e}_1 + (z_{11} + z_{12}\mathbf{i}_1, z_{21} + z_{22}\mathbf{i}_1)_2 \mathbf{e}_2. \quad (6.1)$$

However, this bicomplex scalar product is not *closed* under $\mathbb{C}(\mathbf{i}_1)$. In fact, (\cdot, \cdot) will be *closed* under $\mathbb{C}(\mathbf{i}_1)$ if and only if $(\cdot, \cdot)_1 = (\cdot, \cdot)_2$. From Theorem 9, we obtain the following result.

Theorem 10. *Let \mathbb{T} be the canonical \mathbb{T} -module over the ring of bicomplex numbers with a scalar product (\cdot, \cdot) on $\mathbb{C}(\mathbf{i}_1)$. Let also $w_1 = (z_{11} - z_{12}\mathbf{i}_1)\mathbf{e}_1 + (z_{11} + z_{12}\mathbf{i}_1)\mathbf{e}_2$ and $w_2 = (z_{21} - z_{22}\mathbf{i}_1)\mathbf{e}_1 + (z_{21} + z_{22}\mathbf{i}_1)\mathbf{e}_2$, where $z_{11}, z_{12}, z_{21}, z_{22} \in \mathbb{C}(\mathbf{i}_1)$. If we define*

$$(w_1, w_2) := (z_{11} - z_{12}\mathbf{i}_1, z_{21} - z_{22}\mathbf{i}_1)\mathbf{e}_1 + (z_{11} + z_{12}\mathbf{i}_1, z_{21} + z_{22}\mathbf{i}_1)\mathbf{e}_2, \quad (6.2)$$

then $\{\mathbb{T}, (\cdot, \cdot)\}$ is a bicomplex Hilbert space if and only if $\{\mathbb{C}(\mathbf{i}_1), (\cdot, \cdot)\}$ is a Hilbert space.

As an example, let us consider $\{\mathbb{C}(\mathbf{i}_1), (\cdot, \cdot)\}$ with the canonical scalar product given by

$$\begin{aligned} (z_1, z_2) &= (x_1 + y_1\mathbf{i}_1, x_2 + y_2\mathbf{i}_1) \\ &:= x_1x_2 + y_1y_2. \end{aligned}$$

It is well known that $\{\mathbb{C}(\mathbf{i}_1), (\cdot, \cdot)\}$ is a Hilbert space. Hence, from Theorem 10, $\{\mathbb{T}, (\cdot, \cdot)\}$ is a bicomplex Hilbert space. Moreover, it is easy to see that

$$\|w\| = \|w\|_j = |w|_3 = |w|,$$

i.e. the Euclidean metric of \mathbb{R}^4 .

2. Consider now $M = \mathbb{T}^n$, the n -dimensional module with the canonical \mathbb{T} -basis $\{\hat{e}_i \mid i \in \{1, \dots, n\}\}$, the columns of the identity matrix I_n . For any two elements $\hat{X}, \hat{Y} \in \mathbb{T}^n$ given by $\hat{X} = \sum_{i=1}^n x_i \hat{e}_i$ and $\hat{Y} = \sum_{i=1}^n y_i \hat{e}_i$, we define the bicomplex scalar product as

$$(\hat{X}, \hat{Y}) := (\hat{X}^{\dagger_3})^\top \cdot \hat{Y} = \sum_{i=1}^n x_i^{\dagger_3} y_i \in \mathbb{T}. \quad (6.3)$$

It is now easy to verify that properties 1, 2 and 3 of Definition 2 are trivially satisfied. This bicomplex scalar product also implies that $(\widehat{X}, \widehat{X}) = \sum_{i=1}^n x_i^{\dagger 3} x_i = \sum_{i=1}^n |x_i|_{\mathbf{j}}^2 = \mathbf{e}_1 \sum_{i=1}^n |x_{1i} - x_{2i} \mathbf{i}_1|^2 + \mathbf{e}_2 \sum_{i=1}^n |x_{1i} + x_{2i} \mathbf{i}_1|^2$ where $x_i = x_{1i} + x_{2i} \mathbf{i}_2 = (x_{1i} - x_{2i} \mathbf{i}_1) \mathbf{e}_1 + (x_{1i} + x_{2i} \mathbf{i}_1) \mathbf{e}_2$ for $i \in \{1, \dots, n\}$. Hence, the property 4 of Definition 2 is also satisfied and

$$\|\widehat{X}\| = |(\widehat{X}, \widehat{X})^{\frac{1}{2}}| = \left| \left(\sum_{i=1}^n |x_i|_{\mathbf{j}}^2 \right)^{\frac{1}{2}} \right|. \quad (6.4)$$

In this example, the complex vector space $V = \{\sum_{i=1}^n x_i \widehat{e}_i \mid x_i \in \mathbb{C}(\mathbf{i}_1)\}$ is simply the standard complex vector space isomorphic to \mathbb{C}^n . Moreover, the closure property is satisfied since for $\widehat{X}, \widehat{Y} \in V$ we have $x_i, y_i \in \mathbb{C}(\mathbf{i}_1)$ and $x_i^{\dagger 3} y_i = \overline{x_i} y_i \in \mathbb{C}(\mathbf{i}_1)$ such that equation (6.3) gives an element of $\mathbb{C}(\mathbf{i}_1)$.

7. The Dirac Notation Over M

In this section we introduce the Dirac notation usually used in quantum mechanics. For this we have to define correctly kets and bras over a bicomplex Hilbert space which, we remind, is fundamentally a module.

Let M be a \mathbb{T} -module which is free with the finite \mathbb{T} -basis $\{|m_l\rangle \mid l \in \{1, \dots, n\}\}$. Any element of M will be called a *ket module* or, more simply, a *ket*.

Let us rewrite the definition of the bicomplex scalar product in terms of the ket notation.

Definition 7. Let M be a \mathbb{T} -module which is free with the following finite \mathbb{T} -basis $\{|m_l\rangle \mid l \in \{1, \dots, n\}\}$. With each pair $|\phi\rangle$ and $|\psi\rangle$ in M , taken in this order, we associate a bicomplex number, which is their bicomplex scalar product $(|\phi\rangle, |\psi\rangle)$, and which satisfies the following properties:

1. $(|\phi\rangle, |\psi_1\rangle + |\psi_2\rangle) = (|\phi\rangle, |\psi_1\rangle) + (|\phi\rangle, |\psi_2\rangle)$
2. $(|\phi\rangle, \alpha|\psi\rangle) = \alpha(|\phi\rangle, |\psi\rangle)$, $\forall \alpha \in \mathbb{T}$
3. $(|\phi\rangle, |\psi\rangle) = (|\psi\rangle, |\phi\rangle)^{\dagger 3}$
4. $(|\phi\rangle, |\phi\rangle) = 0 \Leftrightarrow |\phi\rangle = 0$.

Let us now define the dual space M^* .

Definition 8. A linear functional χ is a linear operation which associates a bicomplex number with every ket $|\psi\rangle$:

- 1) $|\psi\rangle \longrightarrow \chi(|\psi\rangle) \in \mathbb{T}$
- 2) $\chi(\lambda_1|\psi_1\rangle + \lambda_2|\psi_2\rangle) = \lambda_1\chi(|\psi_1\rangle) + \lambda_2\chi(|\psi_2\rangle)$, $\lambda_1, \lambda_2 \in \mathbb{T}$.

It can be shown that the set of linear functionals defined on the kets $|\psi\rangle \in M$ constitutes a \mathbb{T} -module space, which is called the dual space of M and which will be symbolized by M^* .

Using this definition of M^* , let us define the bra notation.

Definition 9. Any element of the space M^* is called a bra module or, more simply, a bra. It is symbolized by $\langle \cdot |$.

For example, the bra $\langle \chi |$ designates the bicomplex linear functional χ and we shall henceforth use the notation $\langle \chi | \psi \rangle$ to denote the number obtained by causing the linear functional $\langle \chi | \in M^*$ to act on the ket $|\psi\rangle \in M$:

$$\chi(|\psi\rangle) := \langle \chi | \psi \rangle.$$

The existence of a bicomplex scalar product in M will now enable us to show that we can associate, with every ket $|\phi\rangle \in M$, an element of M^* , which will be denoted by $\langle \phi |$.

The ket $|\phi\rangle$ does indeed enable us to define a linear functional: the one which associates (in a linear way), with each ket $|\psi\rangle \in M$, a bicomplex number which is equal to the scalar product $(|\phi\rangle, |\psi\rangle)$ of $|\psi\rangle$ by $|\phi\rangle$. Let $\langle \phi |$ be this linear functional; it is thus defined by the relation:

$$\langle \phi | \psi \rangle = (|\phi\rangle, |\psi\rangle). \quad (7.1)$$

Therefore, the properties of the bicomplex scalar product can be rewritten as:

1. $\langle \phi | (|\psi_1\rangle + |\psi_2\rangle) = \langle \phi | \psi_1 \rangle + \langle \phi | \psi_2 \rangle$
2. $\langle \phi | \alpha \psi \rangle = \alpha \langle \phi | \psi \rangle, \forall \alpha \in \mathbb{T}$
3. $\langle \phi | \psi \rangle = \langle \psi | \phi \rangle^{\dagger 3}$
4. $\langle \phi | \phi \rangle = 0 \Leftrightarrow |\phi\rangle = 0$.

Now, let us define the corresponding projections for the Dirac notation as follows.

Definition 10. Let $|\psi\rangle, |\phi\rangle \in M$ and $|\chi\rangle \in V$. For $k = 1, 2$, we define:

1. $|\psi_{\mathbf{e}_k}\rangle := P_k(|\psi\rangle) \in V$
2. $\langle \phi_{\mathbf{e}_k} | := P_k(\langle \phi |) : V \longrightarrow \mathbb{C}(\mathbf{i}_1)$, where $|\chi\rangle \mapsto P_k(\langle \phi | \chi \rangle)$.

The first definition gives the projection $|\psi_{\mathbf{e}_k}\rangle$ of the ket $|\psi\rangle$ of M . This is well defined from equation (3.2). However, the second definition is more subtle. In the next two theorems, we show that $\langle \phi_{\mathbf{e}_k} |$ is really the bra associated with the ket $|\phi_{\mathbf{e}_k}\rangle$ in V .

Theorem 11. Let $|\phi\rangle \in M$; then

$$\langle \phi_{\mathbf{e}_k} | \in V^*$$

for $k = 1, 2$.

Proof. Let $\lambda_1, \lambda_2 \in \mathbb{C}(\mathbf{i}_1)$ and $|\psi_1\rangle, |\psi_2\rangle \in V$, then

$$\begin{aligned} \langle \phi_{\mathbf{e}_k} | (\lambda_1 |\psi_1\rangle + \lambda_2 |\psi_2\rangle) \rangle &= P_k \left(\langle \phi | (\lambda_1 |\psi_1\rangle + \lambda_2 |\psi_2\rangle) \rangle \right) \\ &= P_k \left(\lambda_1 \langle \phi | \psi_1 \rangle + \lambda_2 \langle \phi | \psi_2 \rangle \right) \\ &= \lambda_1 P_k \left(\langle \phi | \psi_1 \rangle \right) + \lambda_2 P_k \left(\langle \phi | \psi_2 \rangle \right) \\ &= \lambda_1 \langle \phi_{\mathbf{e}_k} | (|\psi_1\rangle) \rangle + \lambda_2 \langle \phi_{\mathbf{e}_k} | (|\psi_2\rangle) \rangle \end{aligned}$$

for $k = 1, 2$.

We will now show that the functional $\langle \phi_{\mathbf{e}_k} |$ can be obtained from the ket $|\phi_{\mathbf{e}_k}\rangle$.

Theorem 12. *Let $|\phi\rangle \in M$ and $|\psi\rangle \in V$; then*

$$\langle \phi_{\mathbf{e}_k} | (|\psi\rangle) \rangle = \langle \phi_{\mathbf{e}_k} | \psi \rangle \quad (7.2)$$

for $k = 1, 2$.

Proof. Using (4.4) in Theorem 4 and the fact that $P_k(|\psi\rangle) = |\psi\rangle$, we obtain

$$\begin{aligned} \langle \phi_{\mathbf{e}_k} | (|\psi\rangle) \rangle &= P_k \left(\langle \phi | \psi \rangle \right) \\ &= P_k \left((|\phi\rangle, |\psi\rangle) \right) \\ &= \left(P_k(|\phi\rangle), P_k(|\psi\rangle) \right) \\ &= \left(P_k(|\phi\rangle), |\psi\rangle \right) \\ &= \left(|\phi_{\mathbf{e}_k}\rangle, |\psi\rangle \right) \\ &= \langle \phi_{\mathbf{e}_k} | \psi \rangle \end{aligned}$$

for $k = 1, 2$.

Corollary 5. *Let $|\phi\rangle, |\psi\rangle \in M$; then*

$$\langle \phi_{\mathbf{e}_k} | \psi_{\mathbf{e}_k} \rangle = \langle \phi | \psi \rangle_{\mathbf{e}_k} \quad (7.3)$$

for $k = 1, 2$.

Proof. From Theorem 12 and the properties of the projectors P_k , we obtain

$$\begin{aligned} \langle \phi_{\mathbf{e}_k} | \psi_{\mathbf{e}_k} \rangle &= P_k \left(\langle \phi | \psi_{\mathbf{e}_k} \rangle \right) \\ &= P_k \left(\mathbf{e}_1 \langle \phi | \psi_{\mathbf{e}_1} \rangle + \mathbf{e}_2 \langle \phi | \psi_{\mathbf{e}_2} \rangle \right) \\ &= P_k \left(\langle \phi | (\mathbf{e}_1 |\psi_{\mathbf{e}_1}\rangle + \mathbf{e}_2 |\psi_{\mathbf{e}_2}\rangle) \rangle \right) \\ &= P_k \left(\langle \phi | \psi \rangle \right) \\ &= \langle \phi | \psi \rangle_{\mathbf{e}_k} \end{aligned}$$

for $k = 1, 2$.

