On a Tricomplex Distance Estimation for Generalized Multibrot Sets

Dominic Rochon[1]

Joint work with Pierre-Olivier Parisé
Département de mathématiques et d'informatique
Université du Québec, Trois-Rivières

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Bicomplex Numbers

Definition 1 ($\mathbb{M}(2)$ or $\mathbb{BC}$-space)

Let $z_1 = x_1 + x_2\mathbf{i_1}$, $z_2 = x_3 + x_4\mathbf{i_1}$ be two complex numbers $\mathbb{M}(1) \simeq \mathbb{C}$ with $\mathbf{i_1}^2 = -1$. A **bicomplex number** ζ is defined as:

$$\zeta = z_1 + z_2\mathbf{i_2} \tag{1}$$

where $\mathbf{i_2}^2 = -1$.

Various representations:

- In terms of four real numbers: $\zeta = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{j_1}$
- In terms of two idempotent elements:

$$\zeta = (z_1 - z_2\mathbf{i_1})\gamma_1 + (z_1 + z_2\mathbf{i_1})\overline{\gamma}_1$$

where $\gamma_1 = \frac{1+\mathbf{j_1}}{2}$ and $\overline{\gamma}_1 = \frac{1-\mathbf{j_1}}{2}$.

Operations on Bicomplex Numbers

Let $\zeta_1 = z_1 + z_2\mathbf{i_2}$ and $\zeta_2 = z_3 + z_4\mathbf{i_2}$.

1) Equality: $\zeta_1 = \zeta_2 \iff z_1 = z_3$ and $z_2 = z_4$.
2) Addition: $\zeta_1 + \zeta_2 := (z_1 + z_3) + (z_2 + z_4)\mathbf{i_2}$.
3) Multiplication: $\zeta_1 \cdot \zeta_2 := (z_1 z_3 - z_2 z_4) + (z_2 z_3 + z_1 z_4)\mathbf{i_2}$.
4) Euclidean Norm: $|\zeta_1| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\sum_{i=1}^{4} x_i^2}$

Remark:

- $(\mathbb{M}(2), +, \cdot)$ forms a commutative ring with unity and zero divisors.
- $(\mathbb{M}(2), +, \cdot, |\cdot|)$ forms a **Banach space**.

Tricomplex Numbers

Definition 2 ($\mathbb{M}(3)$ or $\mathbb{TC}$-space)

Let $\zeta_1 = z_1 + z_2\mathbf{i_2}$, $\zeta_2 = z_3 + z_4\mathbf{i_2}$ be two bicomplex numbers. A **tricomplex number** η is defined as:

$$\eta = \zeta_1 + \zeta_2\mathbf{i_3} \tag{2}$$

where $\mathbf{i_3}^2 = -1$.

Various representations:

- In terms of four complex numbers: $\eta = z_1 + z_2\mathbf{i_2} + z_3\mathbf{i_3} + z_4\mathbf{j_3}$
- In terms of eight real numbers:

$$\eta = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{i_3} + x_5\mathbf{i_4} + x_6\mathbf{j_1} + x_7\mathbf{j_2} + x_8\mathbf{j_3}$$

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Tricomplex Numbers

Various representations (continuing):

- In terms of two idempotent elements:

$$\eta = (\zeta_1 - \zeta_2 \mathbf{i_2})\gamma_3 + (\zeta_1 + \zeta_2 \mathbf{i_2})\overline{\gamma}_3$$

where $\zeta_1, \zeta_2 \in \mathbb{M}(2)$, $\gamma_3 = \frac{1+\mathbf{j_3}}{2}$ and $\overline{\gamma}_3 = \frac{1-\mathbf{j_3}}{2}$.

- In terms of four idempotent elements:

$$\eta = \eta_{\gamma_1\gamma_3} \cdot \gamma_1\gamma_3 + \eta_{\gamma_1\overline{\gamma}_3} \cdot \gamma_1\overline{\gamma}_3 + \eta_{\overline{\gamma}_1\gamma_3} \cdot \overline{\gamma}_1\gamma_3 + \eta_{\overline{\gamma}_1\overline{\gamma}_3} \cdot \overline{\gamma}_1\overline{\gamma}_3$$

where $\eta_{\gamma_1\gamma_3}, \eta_{\gamma_1\overline{\gamma}_3}, \eta_{\overline{\gamma}_1\gamma_3}, \eta_{\overline{\gamma}_1\overline{\gamma}_3} \in \mathbb{M}(1) \simeq \mathbb{C}$ are defined as the **projections** in the plane.

Subsets of $\mathbb{M}(3)$

Definition 3

Let $\mathbf{i_k} \in \{\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}\}$ and $\mathbf{j_k} \in \{\mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}\}$, where $\mathbf{i_k}^2 = -\mathbf{1}$ and $\mathbf{j_k}^2 = \mathbf{1}$. We define

$$\mathbb{C}(\mathbf{i_k}) := \{\eta = x_0 + x_1\mathbf{i_k} \,:\, x_0, x_1 \in \mathbb{R}\}$$

and

$$\mathbb{D}(\mathbf{j_k}) := \{x_0 + x_1\mathbf{j_k} \,:\, x_0, x_1 \in \mathbb{R}\}.$$

- $\mathbb{C}(\mathbf{i_k})$ is a subset of $\mathbb{M}(3)$ for $k \in \{1, 2, 3, 4\}$. They are all isomorphic to $\mathbb{C}$. Notice that $\mathbb{C}(\mathbf{i_1}) = \mathbb{M}(1)$.
- $\mathbb{D}(\mathbf{j_k})$ is a subset of $\mathbb{M}(3)$ and is isomorphic to the set of hyperbolic numbers $\mathbb{D}$ for $k \in \{1, 2, 3\}$.

Subsets of $\mathbb{M}(3)$ (continuing)

Definition 4

Let $\mathbf{i_k}, \mathbf{i_l}, \mathbf{i_m} \in \{1, \mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}, \mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}\}$ with $\mathbf{i_k} \neq \mathbf{i_l}$, $\mathbf{i_k} \neq \mathbf{i_m}$ and $\mathbf{i_l} \neq \mathbf{i_m}$. The third subset is

$$\mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) := \{x_1\mathbf{i_m} + x_2\mathbf{i_k} + x_3\mathbf{i_l} \,:\, x_1, x_2, x_3 \in \mathbb{R}\}. \tag{3}$$

- $\mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) = \mathrm{span}_{\mathbb{R}}\{\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}\}$.
- This sub-vector space of $\mathbb{M}(3)$ is used to make 3D slices in the tricomplex multibrot sets.

Definition of Multibrots in the complex plane

Definition 5

Let $Q_{p,c}(z) = z^p + c$ a polynomial of degree $p \in \mathbb{N} \setminus \{0, 1\}$. A *Multibrot* set is the set of complex numbers c for which the sequence $\left\{Q_{p,c}^m(0)\right\}_{m=1}^{\infty}$ is bounded, *i.e.*

$$\mathcal{M}^p = \left\{ c \in \mathbb{C} : \left\{Q_{p,c}^m(0)\right\}_{m=1}^{\infty} \text{ is bounded} \right\}. \tag{4}$$

- If we set $p = 2$, we find the well-known Mandelbrot set.

Properties of Multibrot sets

Theorem 6

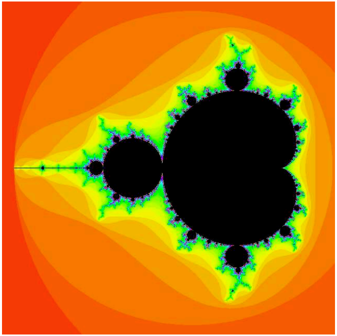
For all complex number c in $\mathcal{M}^p$, we have $|c| \leq 2^{1/(p-1)}$.

Theorem 7

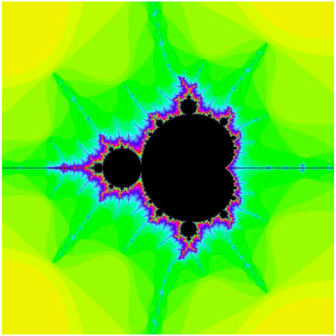
A complex number c is in $\mathcal{M}^p$ if and only if $|Q_{p,c}^m(0)| \leq 2^{1/(p-1)}$ for all natural number $m \geq 1$.

- For an integer $p \geq 2$, the set $\mathcal{M}^p$ is contained a closed discus in $\mathbb{C}$.
- Theorem 7 gives a method to visualize the Multibrot sets.

Multibrot sets Pictured

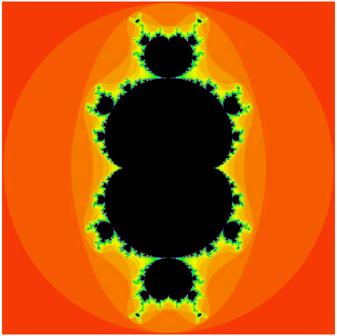


(a) $\mathcal{M}^2$: Mandelbrot set

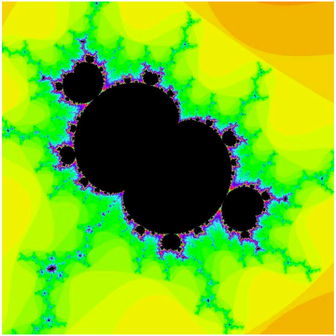


(b) $\mathcal{M}^2$: Zoom in

Multibrot sets Pictured

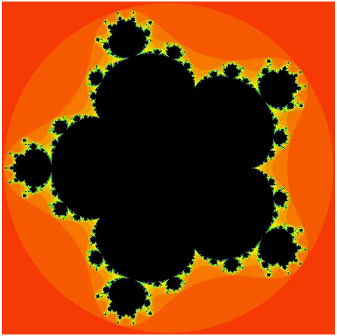


(a) $\mathcal{M}^3$: Mandelbric set

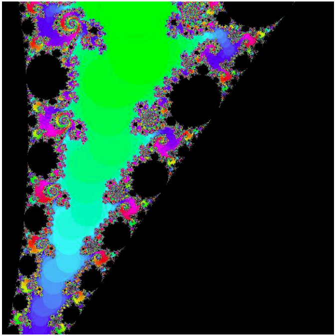


(b) $\mathcal{M}^3$: Zoom in

Multibrot sets Pictured



(a) $\mathcal{M}^6$



(b) $\mathcal{M}^6$: Zoom in

Tricomplex Multibrot Sets

Definition 8

Let $Q_{p,c}(\eta) = \eta^p + c$ where $\eta, c \in \mathbb{M}(3)$ and $p \geq 2$ an integer. The **tricomplex Multibrot** of order p is define as the set

$$\mathcal{M}_3^p := \left\{ c \in \mathbb{M}(3) \ : \ \left\{ Q_{p,c}^m(0) \right\}_{m=1}^{\infty} \text{ is bounded} \right\}. \tag{5}$$

Theorem 9

A tricomplex number c is in $\mathcal{M}_3^p$ if and only if $|Q_{p,c}^m(0)| \leq 2^{1/(p-1)}$ for all natural number $m \geq 1$.

Tricomplex Multibrot Sets

From the idempotent representations, we can define the $\mathbb{TC}$-Cartesian product as

$$X_1 \times_{\gamma_3} X_2 := \{x_1\gamma_3 + x_2\overline{\gamma}_3 \ : \ x_1 \in X_1,\ x_2 \in X_2\}$$

where $X_1, X_2 \subset \mathbb{BC}$. Moreover, we have the following $\mathbb{BC}$-Cartesian product

$$X_1 \times_{\gamma_1} X_2 := \{x_1\gamma_1 + x_2\overline{\gamma}_1 \ : \ x_1 \in X_1,\ x_2 \in X_2\}$$

where $X_1, X_2 \subset \mathbb{C}(\mathbf{i_1})$.

Theorem 10

$$\mathcal{M}_3^p = (\mathcal{M}^p \times_{\gamma_1} \mathcal{M}^p) \times_{\gamma_3} (\mathcal{M}^p \times_{\gamma_1} \mathcal{M}^p).$$

3D slices of $\mathcal{M}_3^p$

To visualize the tricomplex multibrot sets, we have to define a **principal 3D slice of** $\mathcal{M}_3^p$.

$$\mathcal{T}^p := \mathcal{T}^p(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) = \left\{ c \in \mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) \ : \ \left\{ Q_{p,c}^m(0) \right\}_{m=1}^{\infty} \text{ is bounded } \right\}.$$

There are 56 possible 3D principal slices.

Equivalence between 3D slices of $\mathcal{M}_3^p$

Definition 11

Let $\mathcal{T}_1^p(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l})$ and $\mathcal{T}_2^p(\mathbf{i_n}, \mathbf{i_q}, \mathbf{i_s})$ be two principal 3D slices of a tricomplex Multibrot set $\mathcal{M}_3^p$. Then, $\mathcal{T}_1^p \sim \mathcal{T}_2^p$ if we have a bijective linear mapping $\varphi : \mathbb{M}(3) \to \mathbb{M}(3)$ such that $\forall c_2 \in \mathbb{T}(\mathbf{i_n}, \mathbf{i_q}, \mathbf{i_s})$ there exists a $c_1 \in \mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l})$ with $\varphi(c_1) = c_2$ and

$$(\varphi \circ Q_{p,c_1} \circ \varphi^{-1})(\eta) = Q_{p,c_2}(\eta) \ \forall \eta \in \mathbb{M}(3).$$

In that case, we say that $\mathcal{T}_1^p$ and $\mathcal{T}_2^p$ have the **same dynamics**.

Principal Slices of $\mathcal{M}_3^2$

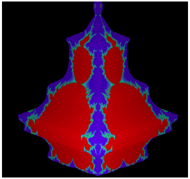
The number of principal 3D slices of the set $\mathcal{M}_3^2$ can be reduced to 8 slices.

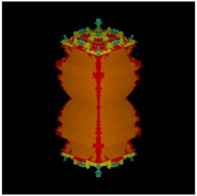
Theorem 12

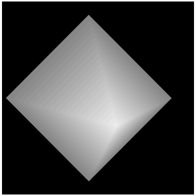
There are eight principal 3D slices of the tricomplex multibrot set $\mathcal{M}_3^2$:

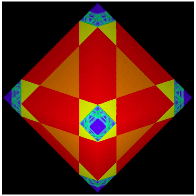
- $\mathcal{T}^2(1, \mathbf{i_1}, \mathbf{i_2})$ *called Tetrabrot;*
- $\mathcal{T}^2(\mathbf{i_1}, \mathbf{j_1}, \mathbf{j_2})$ *called Hourglassbrot;*
- $\mathcal{T}^2(1, \mathbf{j_1}, \mathbf{j_2})$ *called Perplexbrot;*
- $\mathcal{T}^2(\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3})$ *called Metabrot.*
- $\mathcal{T}^2(\mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3})$ *called Firebrot;*
- $\mathcal{T}^2(\mathbf{i_1}, \mathbf{i_2}, \mathbf{j_1})$ *called Mousebrot;*
- $\mathcal{T}^2(\mathbf{i_1}, \mathbf{i_2}, \mathbf{j_2})$ *called Turtlebrot;*
- $\mathcal{T}^2(1, \mathbf{i_1}, \mathbf{j_1})$ *called Arrow-Pitbrot.*

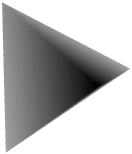
Family Shooting: square $\eta^2 + c$

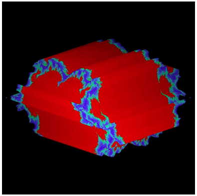

(a) Tetrabrot

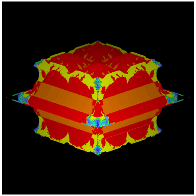

(b) Hourglassbrot

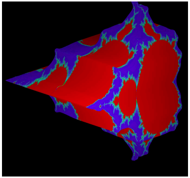

(c) Perplexbrot


(d) Metabrot


(e) Firebrot


(f) Mousebrot


(g) Turtlebrot


(h) Arrow-Pitbrot

Principal Slices of $\mathcal{M}_3^3$

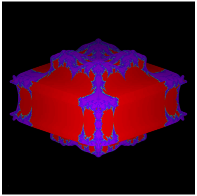
The number of principal 3D slices of the set $\mathcal{M}_3^3$ can be reduced to only 4 slices!

Theorem 13

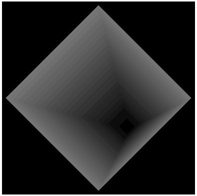
There are four principal 3D slices of the tricomplex multibrot set $\mathcal{M}_3^3$:

- $\mathcal{T}^3(1, \mathbf{i_1}, \mathbf{i_2})$ *called Tetrabric;*
- $\mathcal{T}^3(1, \mathbf{j_1}, \mathbf{j_2})$ *called Perplexbric;*
- $\mathcal{T}^3(1, \mathbf{i_1}, \mathbf{j_1})$ *called Hourglassbric;*
- $\mathcal{T}^3(\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3})$ *called Metabric.*

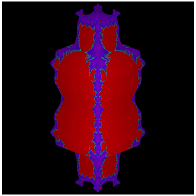
Family Shooting: cubic $\eta^3 + c$



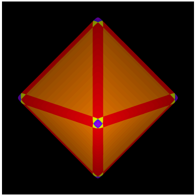
(a) Tetrabric



(b) Perplexbric



(c) Hourglassbric



(d) Metabric

Main Result

To explore these sets deeply, we need to used the ray tracing technique. We have the following theorem for the bounds of the distance from a point $c \in \mathbb{C}(\mathbf{i_1})\backslash\mathcal{M}^p$ to the set $\mathcal{M}^p$.

Theorem 14

Let $c \in \mathbb{C}(\mathbf{i_1})\backslash\mathcal{M}^p$ and define $d(c, \mathcal{M}^p) := \inf\{|z - c| \, : \, z \in \mathcal{M}^p\}$. Then,

$$\frac{\sinh(G(c))}{2e^{G(c)}|G'(c)|} < d(c, \mathcal{M}^p) < \frac{2\sinh(G(c))}{|G'(c)|}$$

where G is the Green's function of the set $\mathcal{M}^p$.

Main Result

We proposed to use this approximation formula, for $p \geq 2$, in the plane.

Conjecture 1

The distance $d(c, \mathcal{M}^p)$ from a point $c \notin \mathcal{M}^p$ to the set $\mathcal{M}^p$ can be approximated in the following way:

$$\frac{p|c_m| \ln |c_m|}{2|c_m|^{1/p^m}|c'_m|} < d(c, \mathcal{M}^p)$$

where $c_m := Q^m_{p,c}(0)$, *and* $c'_m := \frac{d}{dc}\left(Q^m_{p,c}(0)\right)\Big|_{c=c_0}$.

Main Result

In the tricomplex space, the key result is the following. Define, for $\eta_0 \notin X$,

$$d(\eta', X) := \inf \{|\eta - \eta'| \; : \; \eta \in X\}$$

which give the distance between the point η' and the set $X \subset \mathbb{TC}$.

Theorem 15

If $X \subset \mathbb{TC}$ is a compact set, and

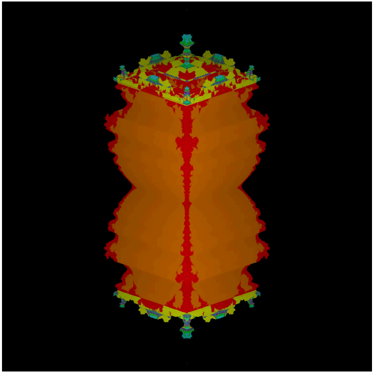
$$X = (X_{\gamma_1\gamma_3} \times_{\gamma_1} X_{\overline{\gamma}_1\gamma_3}) \times_{\gamma_3} (X_{\gamma_1\overline{\gamma}_3} \times_{\gamma_1} X_{\overline{\gamma}_1\overline{\gamma}_3}),$$

then

$$d(\eta', X) = \sqrt{\frac{d(\eta'_{\gamma_1\gamma_3}, X_{\gamma_1\gamma_3})^2 + d(\eta'_{\gamma_1\overline{\gamma}_3}, X_{\gamma_1\overline{\gamma}_3})^2 + d(\eta'_{\overline{\gamma}_1\gamma_3}, X_{\overline{\gamma}_1\gamma_3})^2 + d(\eta'_{\overline{\gamma}_1\overline{\gamma}_3}, X_{\overline{\gamma}_1\overline{\gamma}_3})^2}{4}}.$$

Houglassbrot Exploration

We are now able to use the approximation formula of the complex plane in each idempotent components of a tricomplex multibrot set.



(a) $\mathcal{T}^2(\mathbf{i_1}, \mathbf{j_1}, \mathbf{j_2})$ [YouTube Hyperlink]

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Table of imaginary units

$\cdot$	1	$\mathbf{i_1}$	$\mathbf{i_2}$	$\mathbf{i_3}$	$\mathbf{i_4}$	$\mathbf{j_1}$	$\mathbf{j_2}$	$\mathbf{j_3}$
1	1	$\mathbf{i_1}$	$\mathbf{i_2}$	$\mathbf{i_3}$	$\mathbf{i_4}$	$\mathbf{j_1}$	$\mathbf{j_2}$	$\mathbf{j_3}$
$\mathbf{i_1}$	$\mathbf{i_1}$	$-\mathbf{1}$	$\mathbf{j_1}$	$\mathbf{j_2}$	$-\mathbf{j_3}$	$-\mathbf{i_2}$	$-\mathbf{i_3}$	$\mathbf{i_4}$
$\mathbf{i_2}$	$\mathbf{i_2}$	$\mathbf{j_1}$	$-\mathbf{1}$	$\mathbf{j_3}$	$-\mathbf{j_2}$	$-\mathbf{i_1}$	$\mathbf{i_4}$	$-\mathbf{i_3}$
$\mathbf{i_3}$	$\mathbf{i_3}$	$\mathbf{j_2}$	$\mathbf{j_3}$	$-\mathbf{1}$	$-\mathbf{j_1}$	$\mathbf{i_4}$	$-\mathbf{i_1}$	$-\mathbf{i_2}$
$\mathbf{i_4}$	$\mathbf{i_4}$	$-\mathbf{j_3}$	$-\mathbf{j_2}$	$-\mathbf{j_1}$	$-\mathbf{1}$	$\mathbf{i_3}$	$\mathbf{i_2}$	$\mathbf{i_1}$
$\mathbf{j_1}$	$\mathbf{j_1}$	$-\mathbf{i_2}$	$-\mathbf{i_1}$	$\mathbf{i_4}$	$\mathbf{i_3}$	$\mathbf{1}$	$-\mathbf{j_3}$	$-\mathbf{j_2}$
$\mathbf{j_2}$	$\mathbf{j_2}$	$-\mathbf{i_3}$	$\mathbf{i_4}$	$-\mathbf{i_1}$	$\mathbf{i_2}$	$-\mathbf{j_3}$	$\mathbf{1}$	$-\mathbf{j_1}$
$\mathbf{j_3}$	$\mathbf{j_3}$	$\mathbf{i_4}$	$-\mathbf{i_3}$	$-\mathbf{i_2}$	$\mathbf{i_1}$	$-\mathbf{j_2}$	$-\mathbf{j_1}$	$\mathbf{1}$

Table: Product of tricomplex imaginary units

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