# Platonic Solids and Fractals

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CHAOS 2018, Roma
June 2018

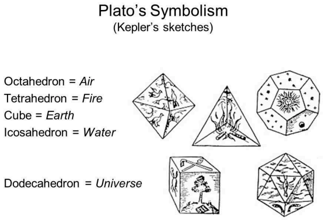
[1]Research supported by CRSNG (Canada).

# Plato's Cosmology

(a) Platonic Solids

> Their mystique [Platonic Solids] has inspired, besides, some of the most fruitful episodes in the development of mathematics and science.
>
> — **Frank Wilczek**
>
> Nobel Prize in Physics in 2004

# Plato's Cosmology




(a) Platonic Solids

The Timaeus, which was considered by many early Platonists and Medieval Neoplatonists to be Plato's most important work, puts forward Plato's cosmology. In particular, the Timaeus provides a deep connexion between the five famous platonic solids and the fundamental elements of the universe. Using the Mandelbrot algorithm in a unified mathematical theory, we present how to generate, from a hypercomplex dynamical system, the following regular platonic solids:

The cube (**Earth**), the Tetrahedron (**Fire**) and the Octahedron (**Air**).

# Bicomplex Numbers

**Definition 1 ($\mathbb{M}(2)$ or $\mathbb{BC}$-space)**

Let $z_1 = x_1 + x_2\mathbf{i_1}$, $z_2 = x_3 + x_4\mathbf{i_1}$ be two complex numbers $\mathbb{M}(1) \simeq \mathbb{C}$ with $\mathbf{i_1}^2 = -1$. A **bicomplex number** $\zeta$ is defined as:

$$\zeta = z_1 + z_2\mathbf{i_2}$$

where $\mathbf{i_2}^2 = -1$.

Various representations:

- In terms of four real numbers: $\zeta = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{j_1}$
- In terms of two idempotent elements:

$$\zeta = (z_1 - z_2\mathbf{i_1})\gamma_1 + (z_1 + z_2\mathbf{i_1})\overline{\gamma}_1$$

where $\gamma_1 = \frac{1+\mathbf{j_1}}{2}$ and $\overline{\gamma}_1 = \frac{1-\mathbf{j_1}}{2}$.

# Operations on Bicomplex Numbers

Let $\zeta_1 = z_1 + z_2\mathbf{i_2}$ and $\zeta_2 = z_3 + z_4\mathbf{i_2}$.

1) Equality: $\zeta_1 = \zeta_2 \iff z_1 = z_3$ and $z_2 = z_4$
2) Addition: $\zeta_1 + \zeta_2 := (z_1 + z_3) + (z_2 + z_4)\mathbf{i_2}$
3) Multiplication: $\zeta_1 \cdot \zeta_2 := (z_1 z_3 - z_2 z_4) + (z_2 z_3 + z_1 z_4)\mathbf{i_2}$
4) Euclidean Norm: $|\zeta_1| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\sum_{i=1}^{4} x_i^2}$

Remark:

- $(\mathbb{M}(2), +, \cdot)$ forms a commutative ring with unity and zero divisors.
- $(\mathbb{M}(2), +, \cdot, |\cdot|)$ forms a **Banach space**.

# Tricomplex Numbers

## Definition 2 ($\mathbb{M}(3)$ or $\mathbb{TC}$-space)

Let $\zeta_1 = z_1 + z_2\mathbf{i_2}$, $\zeta_2 = z_3 + z_4\mathbf{i_2}$ be two bicomplex numbers. A **tricomplex number** $\eta$ is defined as:

$$\eta = \zeta_1 + \zeta_2\mathbf{i_3}$$

where $\mathbf{i_3}^2 = -1$.

Various representations:

- In terms of four complex numbers: $\eta = z_1 + z_2\mathbf{i_2} + z_3\mathbf{i_3} + z_4\mathbf{j_3}$
- In terms of eight real numbers:

$$\eta = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{i_3} + x_5\mathbf{i_4} + x_6\mathbf{j_1} + x_7\mathbf{j_2} + x_8\mathbf{j_3}$$

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# Tricomplex Numbers

Various representations (continuing):

- In terms of two idempotent elements:

$$\eta = (\zeta_1 - \zeta_2 \mathbf{i_2})\gamma_3 + (\zeta_1 + \zeta_2 \mathbf{i_2})\overline{\gamma}_3$$

where $\zeta_1, \zeta_2 \in \mathbb{M}(2)$, $\gamma_3 = \frac{1+\mathbf{j_3}}{2}$ and $\overline{\gamma}_3 = \frac{1-\mathbf{j_3}}{2}$.

- In terms of four idempotent elements:

$$\eta = \eta_{\gamma_1\gamma_3} \cdot \gamma_1\gamma_3 + \eta_{\gamma_1\overline{\gamma}_3} \cdot \gamma_1\overline{\gamma}_3 + \eta_{\overline{\gamma}_1\gamma_3} \cdot \overline{\gamma}_1\gamma_3 + \eta_{\overline{\gamma}_1\overline{\gamma}_3} \cdot \overline{\gamma}_1\overline{\gamma}_3$$

where $\eta_{\gamma_1\gamma_3}, \eta_{\gamma_1\overline{\gamma}_3}, \eta_{\overline{\gamma}_1\gamma_3}, \eta_{\overline{\gamma}_1\overline{\gamma}_3} \in \mathbb{M}(1) \simeq \mathbb{C}$ are defined as the **projections** in the plane.

# Subsets of $\mathbb{M}(3)$

**Definition 3**

Let $\mathbf{i_k} \in \{\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}\}$ and $\mathbf{j_k} \in \{\mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}\}$, where $\mathbf{i_k}^2 = -\mathbf{1}$ and $\mathbf{j_k}^2 = \mathbf{1}$. We define

$$\mathbb{C}(\mathbf{i_k}) := \{x_0 + x_1\mathbf{i_k} \; : \; x_0, x_1 \in \mathbb{R}\}$$

and

$$\mathbb{D}(\mathbf{j_k}) := \{x_0 + x_1\mathbf{j_k} \; : \; x_0, x_1 \in \mathbb{R}\}.$$

- $\mathbb{C}(\mathbf{i_k})$ is a subset of $\mathbb{M}(3)$ for $k \in \{1, 2, 3, 4\}$. They are all isomorphic to $\mathbb{C}$. Notice that $\mathbb{C}(\mathbf{i_1}) = \mathbb{M}(1)$.
- $\mathbb{D}(\mathbf{j_k})$ is a subset of $\mathbb{M}(3)$ and is isomorphic to the set of hyperbolic numbers $\mathbb{D}$ for $k \in \{1, 2, 3\}$.

# Subsets of $\mathbb{M}(3)$ (continuing)

## Definition 4

Let $\mathbf{i_k}, \mathbf{i_l}, \mathbf{i_m} \in \{1, \mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}, \mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}\}$ with $\mathbf{i_k} \neq \mathbf{i_l}$, $\mathbf{i_k} \neq \mathbf{i_m}$ and $\mathbf{i_l} \neq \mathbf{i_m}$. The third subset is

$$\mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) := \{x_1\mathbf{i_m} + x_2\mathbf{i_k} + x_3\mathbf{i_l} \,:\, x_1, x_2, x_3 \in \mathbb{R}\}.$$

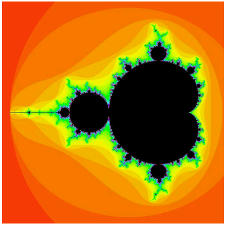
- $\mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) = \text{span}_{\mathbb{R}}\{\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}\}$.
- This sub-vector space of $\mathbb{M}(3)$ is used to make 3D slices in the tricomplex Mandelbrot set.

# The Mandelbrot Set

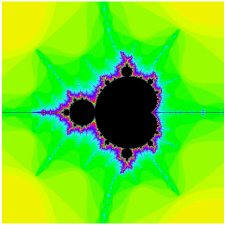
## Definition 5

Let $Q_c(z) = z^2 + c$ a quadratic complex polynomial. The so-called Mandelbrot set is defined as follows:

$$\mathcal{M}^2 = \{c \in \mathbb{C} : \{Q_c^m(0)\}_{m=1}^{\infty} \text{ is bounded } \}.$$



(a) $\mathcal{M}^2$: Mandelbrot set



(b) $\mathcal{M}^2$: Zoom in

# The Hyperbrot Set

In 1990, P. Senn suggested to define the Mandelbrot set for the hyperbolic numbers.

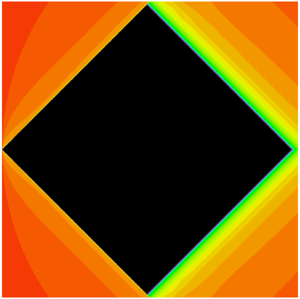
**Definition 6**

Let $Q_c(z) = z^2 + c$ a quadratic hyperbolic polynomial. The *Hyperbrot set* is defined as follows:

$$\mathcal{H}^2 = \{c \in \mathbb{D} : \{Q_c^m(0)\}_{m=1}^{\infty} \text{ is bounded }\}.$$

Seen noticed that the Mandelbrot set for this number structure seemed to be a square. Four years later, a proof of this statement was giving by W. Metzler.

# The Hyperbolic Mandelbrot Set



(a) $\mathcal{H}^2$: Hyperbrot Set

This 2D phenomenon is the fundamental basic tool to obtain polyhedrons with some hypercomplex dynamical systems.

# Tricomplex Mandelbrot Set: The Metatronbrot

## Definition 7

Let $Q_c(\eta) = \eta^2 + c$ where $\eta, c \in \mathbb{M}(3)$. The **tricomplex Mandelbrot set** (also called *Metatronbrot*) is define as the set

$$\mathcal{M}_3^2 := \{c \in \mathbb{M}(3) \ : \ \{Q_c^m(0)\}_{m=1}^{\infty} \text{ is bounded } \}.$$

## Theorem 8

*A tricomplex number $c$ is in $\mathcal{M}_3^2$ if and only if $|Q_c^m(0)| \leq 2$ for all natural number $m \geq 1$.*

# Principal 3D slices of $\mathcal{M}_3^2$

To visualize the 8D tricomplex Mandelbrot set, we have to define a **principal 3D slice of** $\mathcal{M}_3^2$.

$$\mathcal{T}^2 := \mathcal{T}^2(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) = \{ c \in \mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) \ : \ \{Q_c^m(0)\}_{m=1}^{\infty} \text{ is bounded } \}.$$

- There are 56 possible principal 3D slices.
- Any principal 3D slice of the **multicomplex** Mandelbrot set is equivalent to at least one quadricomplex slice or directly to one tricomplex slice up to an affine transformation. Hence, the **tricomplex space** is, in a way, optimal.

# Equivalence between principal 3D slices of $\mathcal{M}_3^2$

## Definition 9

Let $\mathcal{T}_1^2(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l})$ and $\mathcal{T}_2^2(\mathbf{i_n}, \mathbf{i_q}, \mathbf{i_s})$ be two principal 3D slices of the tricomplex Mandelbrot set $\mathcal{M}_3^2$. Then, $\mathcal{T}_1^2 \sim \mathcal{T}_2^2$ if we have a bijective linear mapping $\varphi : M_1 \to M_2$ such that $\varphi(\mathbb{T}_1(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l})) = \mathbb{T}_2(\mathbf{i_n}, \mathbf{i_q}, \mathbf{i_s})$ and, for all $c \in \mathbb{T}_1(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l})$

$$(\varphi \circ Q_c \circ \varphi^{-1})(\eta) = Q_{\varphi(c)}(\eta) \ \forall \eta \in M_2,$$

where $M_i$ is the smallest sub-vector space of $\mathbb{M}(3)$ containing all iterates of $Q_{c_i}^m(0)$ with $c_i \in \mathbb{T}_i$ for $i = 1, 2$. In that case, we say that $\mathcal{T}_1^2$ and $\mathcal{T}_2^2$ have the **same dynamics**.

# Principal Slices of $\mathcal{M}_3^2$

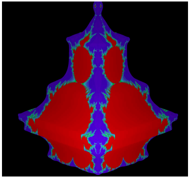
The number of principal 3D slices of the Metatronbrot $\mathcal{M}_3^2$ can be reduced to 8 slices.

**Theorem 10**

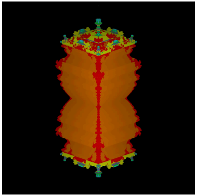
*There are eight **principal 3D slices** of the tricomplex Mandelbrot set $\mathcal{M}_3^2$:*

- $\mathcal{T}^2(1, \mathbf{i_1}, \mathbf{i_2})$ *called Tetrabrot;*
- $\mathcal{T}^2(\mathbf{i_1}, \mathbf{j_1}, \mathbf{j_2})$ *called Hourglassbrot;*
- $\mathcal{T}^2(1, \mathbf{j_1}, \mathbf{j_2})$ *called **Airbrot**;*
- $\mathcal{T}^2(\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3})$ *called Metabrot;*
- $\mathcal{T}^2(\mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3})$ *called **Firebrot**;*
- $\mathcal{T}^2(\mathbf{i_1}, \mathbf{i_2}, \mathbf{j_1})$ *called Mousebrot;*
- $\mathcal{T}^2(\mathbf{i_1}, \mathbf{i_2}, \mathbf{j_2})$ *called Turtlebrot;*
- $\mathcal{T}^2(1, \mathbf{i_1}, \mathbf{j_1})$ *called Arrowheadbrot.*

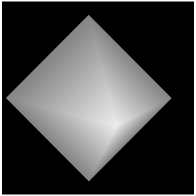
# Family Shooting of the Metatronbrot: $\eta^2 + c$



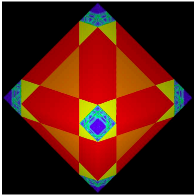
(a) Tetrabrot



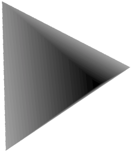
(b) Hourglassbrot



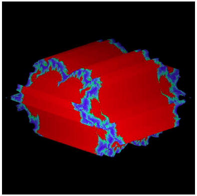
(c) Airbrot



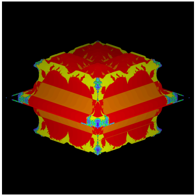
(d) Metabrot



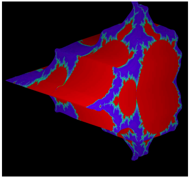
(e) Firebrot



(f) Mousebrot



(g) Turtlebrot



(h) Arrowheadbrot

# Idempotent 3D slices of $\mathcal{M}_3^2$

Using the following idempotent representation of a tricomplex number $\eta$:

$$\eta_{\gamma_1\gamma_3}\cdot\gamma_1\gamma_3+\eta_{\gamma_1\overline{\gamma}_3}\cdot\gamma_1\overline{\gamma}_3+\eta_{\overline{\gamma}_1\gamma_3}\cdot\overline{\gamma}_1\gamma_3+\eta_{\overline{\gamma}_1\overline{\gamma}_3}\cdot\overline{\gamma}_1\overline{\gamma}_3$$

where $\eta_{\gamma_1\gamma_3},\eta_{\gamma_1\overline{\gamma}_3},\eta_{\overline{\gamma}_1\gamma_3},\eta_{\overline{\gamma}_1\overline{\gamma}_3}\in\mathbb{M}(1)\simeq\mathbb{C}$. It is also possible to define some **idempotent 3D slices of $\mathcal{M}_3^2$**. For example, considering

$$c\in\text{span}_{\mathbb{R}}\{\gamma_1\gamma_3,\gamma_1\overline{\gamma}_3,\overline{\gamma}_1\gamma_3\}$$

where each components is related with the dynamics of the real line. Then the set where the polynomial $\{Q_c^m(0)\}_{m=1}^{\infty}$ is bounded is clearly a **cube** in the 3D space $(\gamma_1\gamma_3,\gamma_1\overline{\gamma}_3,\overline{\gamma}_1\gamma_3)$. This specific slice is denoted by

$$\mathcal{T}_e^2(\gamma_1\gamma_3,\gamma_1\overline{\gamma}_3,\overline{\gamma}_1\gamma_3)$$

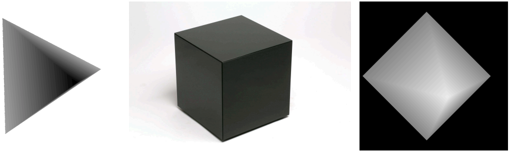
and called the **Earthbrot**.

# Platonic Solids in the Metatronbrot

Here are the **Platonic solids** inside the Metatronbrot as 3D slices of the 8D tricomplex Mandelbrot set.

**Theorem 11**

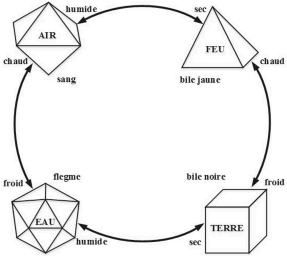
*The* ***Earthbrot*** $\mathcal{T}_e^2(\gamma_1\gamma_3, \gamma_1\overline{\gamma}_3, \overline{\gamma}_1\gamma_3)$ *is a regular cube, the* ***Firebrot*** $\mathcal{T}^2(\mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3)$ *is a regular tetrahedron and the* ***Airbrot*** $\mathcal{T}^2(1, \mathbf{j}_1, \mathbf{j}_2)$ *is a regular octahedron.*



(a) Firebrot (b) Earthbrot (c) Airbrot

NOTE: The name **Metatronbrot** refers to the so-called **Metatron's cube** of the *Flower of Life*, where the same three Platonic solids can be found with an orthogonal projection.

# Aristotelian Elements and Qualities



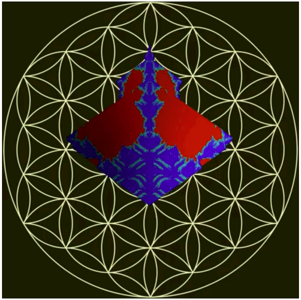

(a) Transformations of the Elements

Aristotle criticizes the theory of his master Plato and explains the change in the material by invoking the qualities associated with the four elements.

**Ref.** Yves Gingras, Histoire des sciences, 2018

# Fractal Alchemy

Using the **Ticomplex Dynamics** we are now able to propose a 3D Fractal Transitions Model between the fundamental elements: Earth, Fire and Air of **Plato's cosmology**.



(a) [Youtube Hyperlink]

NOTE: The Tetrabrot appears at 2:10.

# Thanks for your attention!

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# Table of imaginary units

| $\cdot$ | $1$ | $\mathbf{i_1}$ | $\mathbf{i_2}$ | $\mathbf{i_3}$ | $\mathbf{i_4}$ | $\mathbf{j_1}$ | $\mathbf{j_2}$ | $\mathbf{j_3}$ |
|---|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $\mathbf{i_1}$ | $\mathbf{i_2}$ | $\mathbf{i_3}$ | $\mathbf{i_4}$ | $\mathbf{j_1}$ | $\mathbf{j_2}$ | $\mathbf{j_3}$ |
| $\mathbf{i_1}$ | $\mathbf{i_1}$ | $-\mathbf{1}$ | $\mathbf{j_1}$ | $\mathbf{j_2}$ | $-\mathbf{j_3}$ | $-\mathbf{i_2}$ | $-\mathbf{i_3}$ | $\mathbf{i_4}$ |
| $\mathbf{i_2}$ | $\mathbf{i_2}$ | $\mathbf{j_1}$ | $-\mathbf{1}$ | $\mathbf{j_3}$ | $-\mathbf{j_2}$ | $-\mathbf{i_1}$ | $\mathbf{i_4}$ | $-\mathbf{i_3}$ |
| $\mathbf{i_3}$ | $\mathbf{i_3}$ | $\mathbf{j_2}$ | $\mathbf{j_3}$ | $-\mathbf{1}$ | $-\mathbf{j_1}$ | $\mathbf{i_4}$ | $-\mathbf{i_1}$ | $-\mathbf{i_2}$ |
| $\mathbf{i_4}$ | $\mathbf{i_4}$ | $-\mathbf{j_3}$ | $-\mathbf{j_2}$ | $-\mathbf{j_1}$ | $-\mathbf{1}$ | $\mathbf{i_3}$ | $\mathbf{i_2}$ | $\mathbf{i_1}$ |
| $\mathbf{j_1}$ | $\mathbf{j_1}$ | $-\mathbf{i_2}$ | $-\mathbf{i_1}$ | $\mathbf{i_4}$ | $\mathbf{i_3}$ | $\mathbf{1}$ | $-\mathbf{j_3}$ | $-\mathbf{j_2}$ |
| $\mathbf{j_2}$ | $\mathbf{j_2}$ | $-\mathbf{i_3}$ | $\mathbf{i_4}$ | $-\mathbf{i_1}$ | $\mathbf{i_2}$ | $-\mathbf{j_3}$ | $\mathbf{1}$ | $-\mathbf{j_1}$ |
| $\mathbf{j_3}$ | $\mathbf{j_3}$ | $\mathbf{i_4}$ | $-\mathbf{i_3}$ | $-\mathbf{i_2}$ | $\mathbf{i_1}$ | $-\mathbf{j_2}$ | $-\mathbf{j_1}$ | $\mathbf{1}$ |

Table: Product of tricomplex imaginary units

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