On a generalized Fatou-Julia theorem in multicomplex spaces

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Definition *Multicomplex Numbers*

Multicomplex numbers of order n are defined as

$$\mathbb{M}(n) := \{\zeta_1 + \zeta_2 \mathbf{i_n} \mid \zeta_1, \zeta_2 \in \mathbb{M}(n-1)\} \tag{1}$$

with ${\mathbf{i_n}}^2 = -1$.

- Note that multicomplex addition and multiplication are associative and commutative.
- Multicomplex numbers of order n are a sub-algebra of the Clifford algebra $\mathrm{Cl}_{\mathbb{R}}(0, 2n)$.

Examples

- Bicomplex Numbers (n=2) :

$$\mathbb{M}(2) := \mathbb{T} := \{z_1 + z_2\mathbf{i_2} \mid z_1, z_2 \in \mathbb{C}\}$$

- Tricomplex Numbers (n=3) :

$$\mathbb{M}(3) := \{w_1 + w_2\mathbf{i_3} \mid w_1, w_2 \in \mathbb{M}(2)\}$$

- $n = 1$ and $n = 0$ correspond respectively to $\mathbb{C}$ and $\mathbb{R}$.

Another Notation

Decomposing ζ_1, ζ_2 from (1), we obtain

$$\mathbb{M}(n) := \{\zeta_{11} + \zeta_{12}\mathbf{i_{n-1}} + \zeta_{21}\mathbf{i_n} + \zeta_{22}\mathbf{j_n} \mid \zeta_{11}, \zeta_{12}, \zeta_{21}, \zeta_{22} \in \mathbb{M}(n-2)\} \quad (2)$$

where

$$\mathbf{j_n} = \mathbf{i_n}\mathbf{i_{n-1}} = \mathbf{i_{n-1}}\mathbf{i_n}$$

and thus $\mathbf{j_n}^2 = 1$.

It follows that

- We can rewrite the set $\mathbb{M}(n)$ with 2^n coefficients x_k in $\mathbb{R}$.
- We can rewrite the set $\mathbb{M}(n)$ with 2^{n-k} coefficients in $\mathbb{M}(k)$, $0 \leq k \leq n$.

An Example : Tricomplex Numbers

Let us apply the last remark to the case of tricomplex numbers ($n = 3$) :

Definition

Tricomplex numbers are defined as (extended notation) :

$$\mathbb{M}(3) := \{\zeta = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{i_3} + x_5\mathbf{i_4} + x_6\mathbf{j_1} + x_7\mathbf{j_2} + x_8\mathbf{j_3} \mid x_i \in \mathbb{R}\} \quad (3)$$

with ${\mathbf{i_k}}^2 = -1,\ k = 1, 2, 3, 4$ and ${\mathbf{j_l}}^2 = 1,\ l = 1, 2, 3.$

An Example : Tricomplex Numbers

Multiplication Rule for Tricomplex Imaginary Units

$\cdot$	1	i_1	i_2	i_3	i_4	j_1	j_2	j_3
1	1	i_1	i_2	i_3	i_4	j_1	j_2	j_3
i_1	i_1	-1	j_1	j_2	$-j_3$	$-i_2$	$-i_3$	i_4
i_2	i_2	j_1	-1	j_3	$-j_2$	$-i_1$	i_4	$-i_3$
i_3	i_3	j_2	j_3	-1	$-j_1$	i_4	$-i_1$	$-i_2$
i_4	i_4	$-j_3$	$-j_2$	$-j_1$	-1	i_3	i_2	i_1
j_1	j_1	$-i_2$	$-i_1$	i_4	i_3	1	$-j_3$	$-j_2$
j_2	j_2	i_3	i_4	$-i_1$	i_2	$-j_3$	1	$-j_1$
j_3	j_3	i_4	$-i_3$	$-i_2$	i_1	$-j_2$	$-j_1$	1

Definition

Let the following multicomplex numbers :

$$\gamma_{n-1} := \frac{1 + \mathbf{i_{n-1}} \mathbf{i_n}}{2} = \frac{1 + \mathbf{j_n}}{2}; \quad \overline{\gamma}_{n-1} := \frac{1 - \mathbf{i_{n-1}} \mathbf{i_n}}{2} = \frac{1 - \mathbf{j_n}}{2}; \tag{4}$$

Then we can write the number $\zeta = \zeta_1 + \zeta_2 \mathbf{i_n}$ as :

$$\zeta = (\zeta_1 - \zeta_2 \mathbf{i_{n-1}})\gamma_{n-1} + (\zeta_1 + \zeta_2 \mathbf{i_{n-1}})\overline{\gamma}_{n-1} \tag{5}$$

Properties

The idempotent elements have the following properties :

$$\begin{aligned} \gamma_{n-1}^2 = \gamma_{n-1} \quad &; \quad \overline{\gamma}_{n-1}^2 = \overline{\gamma}_{n-1}; \\ \gamma_{n-1} + \overline{\gamma}_{n-1} = 1 \quad &; \quad \gamma_{n-1}\overline{\gamma}_{n-1} = \overline{\gamma}_{n-1}\gamma_{n-1} = 0. \end{aligned} \tag{6}$$

An Example : Tricomplex Numbers

The tricomplex idempotent elements are :

$$\gamma_2 := \frac{1+\mathbf{i_2}\mathbf{i_3}}{2} = \frac{1+\mathbf{j_3}}{2}; \quad \overline{\gamma}_2 := \frac{1-\mathbf{i_2}\mathbf{i_3}}{2} = \frac{1-\mathbf{j_3}}{2};$$

Thus we can write

$$\zeta = w_1 + w_2\mathbf{i_3} = (w_1 - w_2\mathbf{i_2})\gamma_2 + (w_1 + w_2\mathbf{i_2})\overline{\gamma}_2$$

- The interest of the idempotent representation is that we can add, multiply and divide term-by-term.
- Without this representation the proofs in multicomplex dynamics would be much more complicated.

An Example : Tricomplex Numbers

As the bicomplex numbers $w_1 = z_1 + z_2\mathbf{i_2}$ and $w_2 = z_3 + z_4\mathbf{i_2}$ can also be decompose into the idempotent representation, we obtain :

$$\zeta = w_{\gamma_1\gamma_2} \cdot \gamma_1\gamma_2 + w_{\overline{\gamma}_1\gamma_2} \cdot \overline{\gamma}_1\gamma_2 + w_{\gamma_1\overline{\gamma}_2} \cdot \gamma_1\overline{\gamma}_2 + w_{\overline{\gamma}_1\overline{\gamma}_2} \cdot \overline{\gamma}_1\overline{\gamma}_2 \ .$$

where

$$\begin{aligned} w_{\gamma_1\gamma_2} &:= (z_1 + z_4) - (z_2 - z_3)\mathbf{i_1} \\ w_{\overline{\gamma}_1\gamma_2} &:= (z_1 + z_4) + (z_2 - z_3)\mathbf{i_1} \\ w_{\gamma_1\overline{\gamma}_2} &:= (z_1 - z_4) - (z_2 + z_3)\mathbf{i_1} \\ w_{\overline{\gamma}_1\overline{\gamma}_2} &:= (z_1 - z_4) + (z_2 + z_3)\mathbf{i_1} \end{aligned} \tag{7}$$

- The elements $\gamma_1\gamma_2$, $\overline{\gamma}_1\gamma_2$, $\gamma_1\overline{\gamma}_2$ and $\overline{\gamma}_1\overline{\gamma}_2$ are also idempotent two by two because of the properties in (6) and thus we can add, multiply and divide term-by-term.

Definition *Generalized Mandelbrot Set*

Consider the multicomplex function P_c, $c \in \mathbb{M}(n)$ defined as $P_c(\zeta) = \zeta^2 + c$. Then we define the generalized Mandelbrot set $\mathcal{M}_n$ for multicomplex numbers of order n :

$$\mathcal{M}_n := \{c \in \mathbb{M}(n) \mid \{P_c^{\circ m}(0)\}_{m=1}^{\infty} \text{ is bounded } \}. \tag{8}$$

Theorem

The generalized Mandelbrot set for multicomplex numbers of order n is connected.

Definition

We say that $X \subset \mathbb{M}(n)$ is the $\mathbb{M}(n)$-cartesian product set determined by $X_1, X_2 \in \mathbb{M}(n-1)$ if

$$\begin{aligned} X &:= X_1 \times_{\gamma_{n-1}} X_2 \\ &:= \{\zeta_1 + \zeta_2 \mathbf{i_n} \in \mathbb{M}(n) \mid \zeta_1 + \zeta_2 \mathbf{i_n} = u_1 \gamma_{n-1} + u_2 \overline{\gamma}_{n-1}, \\ &\quad (u_1, u_2) \in X_1 \times X_2\}. \end{aligned} \tag{9}$$

Theorem

$$\mathcal{M}_n = \mathcal{M}_{n-1} \times_{\gamma_{n-1}} \mathcal{M}_{n-1}$$

Example

The bicomplex Mandelbrot set in term of the classic Mandelbrot set :

$$\mathcal{M}_2 = \mathcal{M} \times_{\gamma_1} \mathcal{M}$$

Definition *Generalized Filled-in Julia Sets*

We define the multicomplex filled-in Julia set of order n corresponding to the number $c \in \mathbb{M}(n)$ as :

$$\mathcal{K}_{n,c} := \{\zeta \in \mathbb{M}(n) \mid \{P_c^{\circ m}(\zeta)\}_{m=1}^{\infty} \text{ is bounded } \}. \qquad (10)$$

Theorem

$$\begin{aligned}\mathcal{K}_{n,c} &= \mathcal{K}_{n,(c_1 - c_2\mathbf{i_{n-1}})\gamma_{n-1} + (c_1 + c_2\mathbf{i_{n-1}})\overline{\gamma}_{n-1}} \\ &= \mathcal{K}_{n-1,c_1 - c_2\mathbf{i_{n-1}}} \times_{\gamma_{n-1}} \mathcal{K}_{n-1,c_1 + c_2\mathbf{i_{n-1}}}.\end{aligned}$$

Theorem

$c \in \mathcal{M}_n \Leftrightarrow \mathcal{K}_{n,c}$ is connected.

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Theorem

$$c \in \mathcal{M}_n \Leftrightarrow \mathcal{K}_{n,c} \text{ is connected.}$$

An Example : Tricomplex Numbers

The Tricomplex Filled Julia Set for the parameter c in term of the Bicomplex and Complex Filled Julia Sets Corresponding

$$\begin{aligned}
\mathcal{K}_{3,c} &= \mathcal{K}_{3,(c_1-c_2\mathbf{i_2})\gamma_2+(c_1+c_2\mathbf{i_2})\overline{\gamma}_2} \\
&= \mathcal{K}_{2,c_1-c_2\mathbf{i_2}} \times_{\gamma_2} \mathcal{K}_{2,c_1+c_2\mathbf{i_2}}. \\
&= \left(\mathcal{K}_{w_{\gamma_1\gamma_2}} \times_{\gamma_1} \mathcal{K}_{w_{\overline{\gamma}_1\gamma_2}}\right) \times_{\gamma_2} \left(\mathcal{K}_{w_{\gamma_1\overline{\gamma}_2}} \times_{\gamma_1} \mathcal{K}_{w_{\overline{\gamma}_1\overline{\gamma}_2}}\right) \qquad (11)
\end{aligned}$$

Definition

Let $\mathcal{K}_{n,c}$ be a multicomplex filled Julia set of the quadratic form $P_c(\zeta) = \zeta^2 + c$. We define the set $A_{n,c}(\infty) := \mathbb{M}(n) \backslash \mathcal{K}_{n,c}$ as the basin of attraction to infinity of $P_c(\zeta)$. We have

$$A_{n,c}(\infty) := \{\zeta \in \mathbb{M}(n) \mid \{P_c^{\circ m}(\zeta)\} \to \infty\}.$$

Definition

Let $SA_{2,c'}(\infty)$ be the strong basin of attraction to infinity of $P_{c'}(w) = w^2 + c'$ defined as

$$SA_{2,c'}(\infty) := A_{c_1'-c_2'\mathbf{i_1}}(\infty) \times_{\gamma_1} A_{c_1'+c_2'\mathbf{i_1}}(\infty)$$

Then we define for $n > 2$

$$SA_{n,c}(\infty) \quad := \quad SA_{n-1,c_1-c_2\mathbf{i_{n-1}}}(\infty) \times_{\gamma_{n-1}} SA_{n-1,c_1+c_2\mathbf{i_{n-1}}}(\infty)$$

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Definition

Let $SA_{2,c'}(\infty)$ be the strong basin of attraction to infinity of $P_{c'}(w) = w^2 + c'$ defined as

$$SA_{2,c'}(\infty) := A_{c_1' - c_2' \mathbf{i_1}}(\infty) \times_{\gamma_1} A_{c_1' + c_2' \mathbf{i_1}}(\infty)$$

Then we define for $n > 2$

$$SA_{n,c}(\infty) \quad := \quad SA_{n-1, c_1 - c_2 \mathbf{i_{n-1}}}(\infty) \times_{\gamma_{n-1}} SA_{n-1, c_1 + c_2 \mathbf{i_{n-1}}}(\infty)$$

as the strong basin of attraction to infinity of $P_c(\zeta) = \zeta^2 + c$.

Theorem *Fatou-Julia*

Let $\mathcal{K}_{n,c}$ be a multicomplex filled Julia set of the quadratic form $P_c(\zeta) = \zeta^2 + c$ where $c \in \mathbb{M}(n)$ and $n \geq 2$. Then

1. $0 \in \mathcal{K}_{n,c} \Leftrightarrow \mathcal{K}_{n,c}$ is connected;
2. $0 \in SA_{n,c}(\infty) \Leftrightarrow \mathcal{K}_{n,c}$ is a Cantor set in $\mathbb{M}(n)$;
3. $0 \in A_{n,c}(\infty) \backslash SA_{n,c}(\infty) \Leftrightarrow \mathcal{K}_{n,c}$ is disconnected but not totally.

An Example : Tricomplex Numbers

As

$$\mathcal{K}_{3,c} = (\mathcal{K}_{w_{\gamma_1\gamma_2}} \times_{\gamma_1} \mathcal{K}_{w_{\overline{\gamma}_1\gamma_2}}) \times_{\gamma_2} (\mathcal{K}_{w_{\gamma_1\overline{\gamma}_2}} \times_{\gamma_1} \mathcal{K}_{w_{\overline{\gamma}_1\overline{\gamma}_2}})$$

we note that :

1. $\mathcal{K}_{3,c}$ is connected if and only if $\mathcal{K}_{w_{\gamma_1\gamma_2}}, \mathcal{K}_{w_{\overline{\gamma}_1\gamma_2}}, \mathcal{K}_{w_{\gamma_1\overline{\gamma}_2}}$ and $\mathcal{K}_{w_{\overline{\gamma}_1\overline{\gamma}_2}}$ are all connected,
2. $\mathcal{K}_{3,c}$ is a Cantor set if and only if $\mathcal{K}_{w_{\gamma_1\gamma_2}}, \mathcal{K}_{w_{\overline{\gamma}_1\gamma_2}}, \mathcal{K}_{w_{\gamma_1\overline{\gamma}_2}}$ and $\mathcal{K}_{w_{\overline{\gamma}_1\overline{\gamma}_2}}$ are all Cantor sets,
3. $\mathcal{K}_{3,c}$ is disconnected but not totally in all other cases.

- From the case 3, we notice that there are 3 subcases : exactly if 1, 2 or 3 of the filled Julia sets in $\mathbb{M}(1)$ are connected.
- The set $\mathcal{K}_{3,c}$ will be "more connected" if there are 3 that are connected than if there is only 1.

An Example : Tricomplex Numbers

As

$$\mathcal{K}_{3,c} = (\mathcal{K}_{w_{\gamma_1\gamma_2}} \times_{\gamma_1} \mathcal{K}_{w_{\overline{\gamma}_1\gamma_2}}) \times_{\gamma_2} (\mathcal{K}_{w_{\gamma_1\overline{\gamma}_2}} \times_{\gamma_1} \mathcal{K}_{w_{\overline{\gamma}_1\overline{\gamma}_2}})$$

we note that :

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- From the case 3, we notice that there are 3 subcases : exactly if 1, 2 or 3 of the filled Julia sets in $\mathbb{M}(1)$ are connected.
- The set $\mathcal{K}_{3,c}$ will be “more connected” if there are 3 that are connected than if there is only 1.

A Link Between $\mathcal{M}_3$ and $\mathcal{K}_{3,c}$

- The c's that correspond to a connected filled Julia set are inside $\mathcal{M}_3$.
- The c's that correspond to a Cantor filled Julia set are on the fractal part of $\mathcal{M}_3$ (Graded pink part).
- The c's that correspond to the three kind of disconnected but not totally filled Julia sets are in three different divergence layers of $\mathcal{M}_3$, the "most connected" closer to the set itself (From most to less connected : green, blue, black).

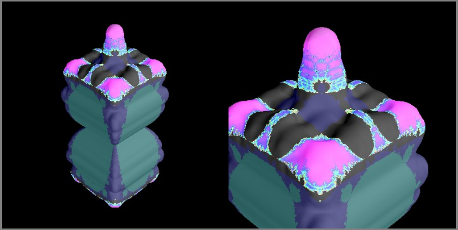

Figure: A Slice of $\mathcal{M}_3$: $\mathcal{T}(\mathbf{i_1}, \mathbf{j_1}, \mathbf{j_2})$

As we wish to visualize $\mathcal{M}_3$, that is in fact in dimension 8, we need to fix 5 of the 8 real coefficients of the tricomplex numbers. Doing that we can see particular 3D slices of the set. Fixing the 5 coefficients to 0 will give the *principal 3D slices* of $\mathcal{M}_3$.

Some Particular Tricomplex Subspaces

Definition

$$\begin{aligned}\mathbb{M}(\mathbf{i_k},\mathbf{i_l}) &:= \{x_1 + x_2\mathbf{i_k} + x_3\mathbf{i_l} + x_4\mathbf{i_k}\mathbf{i_l} \mid \mathbf{i_k},\mathbf{i_l} \in \{\mathbf{i_1},\mathbf{i_2},\mathbf{i_3},\mathbf{i_4},\mathbf{j_1},\mathbf{j_2},\mathbf{j_3}\},\\ &\quad \mathbf{i_k} \neq \mathbf{i_l}, x_1, x_2, x_3, x_4 \in \mathbb{R}\} \qquad (12)\end{aligned}$$

- The sets $\mathbb{M}(\mathbf{i_k},\mathbf{i_l})$ are all closed under multiplication.
- We have $\mathbb{M}(\mathbf{i_k},\mathbf{i_l}) \cong \mathbb{M}(2)$ except for $\mathbb{M}(\mathbf{j_1},\mathbf{j_2})$, $\mathbb{M}(\mathbf{j_1},\mathbf{j_3})$ and $\mathbb{M}(\mathbf{j_2},\mathbf{j_3})$, which are all the same and so we shall call it the **biduplex** set and note it $\mathbb{D}(2)$.

Some Particular Tricomplex Subspaces

Definition

$$\mathbb{T}(\mathbf{i_k},\mathbf{i_l},\mathbf{i_m}) := \{x_1\mathbf{i_k}+x_2\mathbf{i_l}+x_3\mathbf{i_m} \mid \mathbf{i_k},\mathbf{i_l},\mathbf{i_m}\in\{1,\mathbf{i_1},\mathbf{i_2},\mathbf{i_3},\mathbf{i_4},\mathbf{j_1},\mathbf{j_2},\mathbf{i_2}\}, \mathbf{i_k}\neq\mathbf{i_l}\neq\mathbf{i_m}; x_1,x_2,x_3\in\mathbb{R}\} \quad (13)$$

- The sets $\mathbb{T}(\mathbf{i_k},\mathbf{i_l},\mathbf{i_m})$ are not closed under multiplication.
- We have $\mathbb{T}(\mathbf{i_k},\mathbf{i_l},\mathbf{i_m})\subset\mathbb{M}(3)$ and for some $\mathbf{i_k},\mathbf{i_l},\mathbf{i_m}\in\{1,\mathbf{i_1},\mathbf{i_2},\mathbf{i_3},\mathbf{i_4},\mathbf{j_1},\mathbf{j_2},\mathbf{i_2}\}, \mathbf{i_k}\neq\mathbf{i_l}\neq\mathbf{i_m}$, we have $\mathbb{T}(\mathbf{i_k},\mathbf{i_l},\mathbf{i_m})\subset\mathbb{M}(\mathbf{i_k},\mathbf{i_l})$.

Definition *Principal 3D Slices of $\mathcal{M}_3$*

The principal 3D slice of $\mathcal{M}_3$ corresponding to $\mathbf{i_k}, \mathbf{i_l}, \mathbf{i_m}$ is defined as

$$\mathcal{T}(\mathbf{i_k}, \mathbf{i_l}, \mathbf{i_m}) := \{c \in \mathbb{T}(\mathbf{i_k}, \mathbf{i_l}, \mathbf{i_m}) \mid \{P_c^{\circ n}(0)\}_{n=1}^{\infty} \text{ is bounded}\} \tag{14}$$

Example : $\mathbf{i_k} = 1, \mathbf{i_l} = \mathbf{i_1}, \mathbf{i_m} = \mathbf{i_2}$

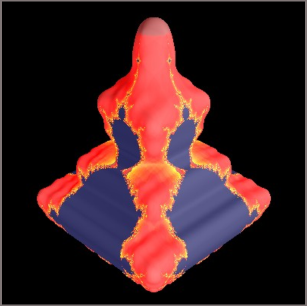

Figure: The Classical Tetrabrot, $\mathcal{T}(1, \mathbf{i_1}, \mathbf{i_2})$

Definition

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be 3D slices of $\mathcal{M}_3$ corresponding respectively to the functions P_{c_1} and P_{c_2}. We say that we have $\mathcal{T}_1 \sim \mathcal{T}_2$ if there exist a function φ such that $(\varphi \circ P_{c_1} \circ \varphi^{-1})(\zeta) = P_{c_2}(\zeta)$.

Remarks

- The sets $\mathcal{T}_1$ and $\mathcal{T}_2$ are said to have the same dynamics.
- $\sim$ is an equivalence relation.
- Two sets with the same dynamics will appear exactly the same in a 3D Visualization Software, that is why we will say that they are symmetrical.

Example : The Classical Tetrabrot

For the classical Tetrabrot, we have the following symmetries : $\mathcal{T}(1, \mathbf{i_1}, \mathbf{i_2}) \sim \mathcal{T}(1, \mathbf{i_k}, \mathbf{i_l}), \forall\, \mathbf{i_k}, \mathbf{i_l} \in \{\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}\},\ \mathbf{i_k} \neq \mathbf{i_l}$

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Slice no.2

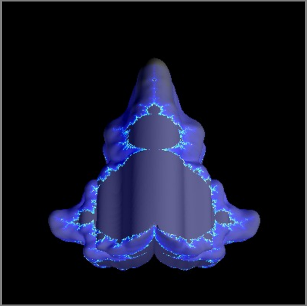

Figure: $\mathcal{T}(1, \mathbf{i_1}, \mathbf{j_1})$

Symmetries for the Slice no.2

$$\mathcal{T}(1, \mathbf{i_1}, \mathbf{j_1}) \sim \mathcal{T}(1, \mathbf{i_k}, \mathbf{i_l}), \forall\ \mathbf{i_k} \in \{\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}\}, \mathbf{i_l} \in \{\mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}\}$$

Slice no.3

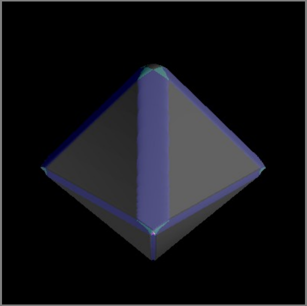

Figure: The Perplexbrot, $\mathcal{T}(1, \mathbf{j}_1, \mathbf{j}_2)$

Symmetries for the Slice no.3

$$\mathcal{T}(1, \mathbf{j}_1, \mathbf{j}_2) \sim \mathcal{T}(1, \mathbf{j}_1, \mathbf{j}_3) \sim \mathcal{T}(1, \mathbf{j}_2, \mathbf{j}_3)$$

Slice no.3

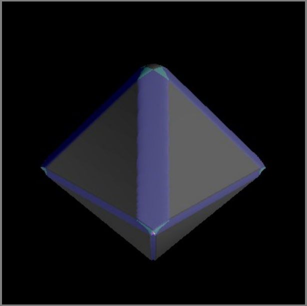

Figure: The Perplexbrot, $\mathcal{T}(1, \mathbf{j_1}, \mathbf{j_2})$

Remarks

- The Perplexbrot can be view as a generalization of the hyperbolic Mandelbrot set.
- It is a regular octahedron of edge length equal to $\frac{9}{8}\sqrt{2}$.

Slice no.4

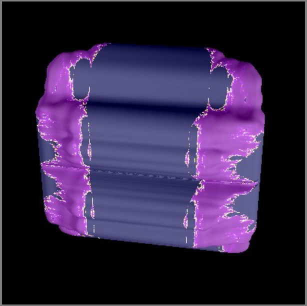

Figure: $\mathcal{T}(\mathbf{i_1}, \mathbf{i_2}, \mathbf{j_1})$

Symmetries for the Slice no.4

$$\mathcal{T}(\mathbf{i_1}, \mathbf{i_2}, \mathbf{j_1}) \sim \mathcal{T}(\mathbf{i_k}, \mathbf{i_l}, \mathbf{i_k}\mathbf{i_l}); \quad \mathbf{i_k}, \mathbf{i_l} \in \{\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}\}, \mathbf{i_k} \neq \mathbf{i_l}.$$

Slice no.5

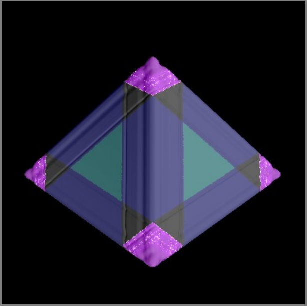

Figure: $\mathcal{T}(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$

Symmetries for the Slice no.5

$$\mathcal{T}(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3) \sim \mathcal{T}(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_4) \sim \mathcal{T}(\mathbf{i}_1, \mathbf{i}_3, \mathbf{i}_4) \sim \mathcal{T}(\mathbf{i}_2, \mathbf{i}_3, \mathbf{i}_4)$$

Slice no.6

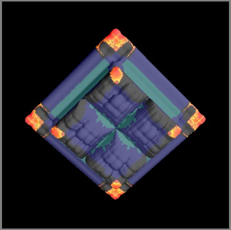

Figure: $\mathcal{T}(\mathbf{i_1}, \mathbf{i_2}, \mathbf{j_2})$

Symmetries for the Slice no.6

$$\mathcal{T}(\mathbf{i_1}, \mathbf{i_2}, \mathbf{j_2}) \sim \mathcal{T}(\mathbf{i_k}, \mathbf{i_l}, \mathbf{i_m}); \quad \mathbf{i_k}, \mathbf{i_l} \in \{\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}\}, \mathbf{i_k} \neq \mathbf{i_l}$$
$$\mathbf{i_m} \in \{\mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}\} \backslash \{\mathbf{i_k} \mathbf{i_l}\}.$$

Slice no.7

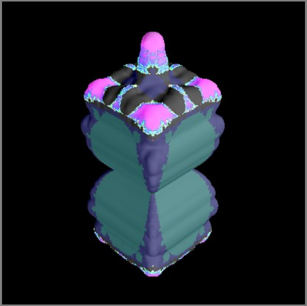

Figure: $\mathcal{T}(\mathbf{i_1}, \mathbf{j_1}, \mathbf{j_2})$

Symmetries for the Slice no.7

$$\mathcal{T}(\mathbf{i_1}, \mathbf{j_1}, \mathbf{j_2}) \sim \mathcal{T}(\mathbf{i_k}, \mathbf{i_l}, \mathbf{i_m}); \quad \mathbf{i_k} \in \{\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}\}, \mathbf{i_l}, \mathbf{i_m} \in \{\mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}\}, \mathbf{i_l} \neq \mathbf{i_m}.$$

Slice no.8

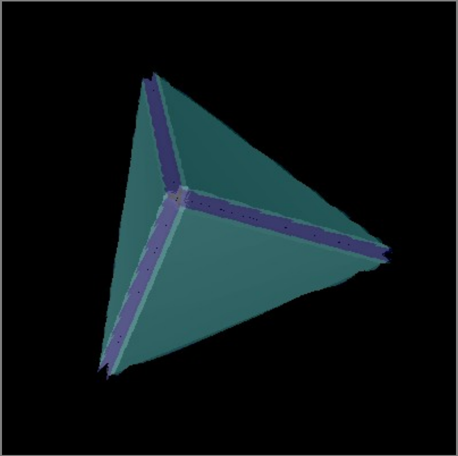

Figure: $\mathcal{T}(\mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3)$

Slice no.8

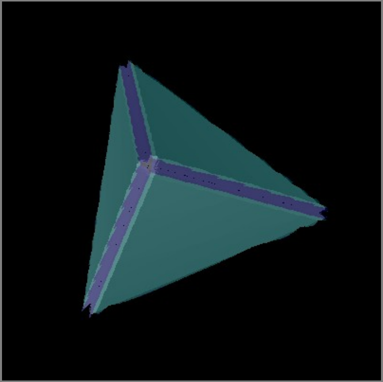

Figure: $\mathcal{T}(\mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3)$

Thank you for your attention!