On a relation of bicomplex pseudoanalytic function theory to the complexified stationary Schrödinger equation

Dominic Rochon, Ph.D.

Département de Mathématiques et d'Informatique
Université du Québec à Trois-Rivières

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Definition

Bicomplex numbers are defined as

$$\mathbb{T} := \{z_1 + z_2\mathbf{i_2} \mid z_1, z_2 \in \mathbb{C}(\mathbf{i_1})\} \tag{1}$$

where the imaginary units $\mathbf{i_1}, \mathbf{i_2}$ and $\mathbf{j}$ are governed by the rules: $\mathbf{i_1}^2 = \mathbf{i_2}^2 = -1$, $\mathbf{j}^2 = 1$ and

$$\begin{array}{rclcl} \mathbf{i_1}\mathbf{i_2} &=& \mathbf{i_2}\mathbf{i_1} &=& \mathbf{j}, \\ \mathbf{i_1}\mathbf{j} &=& \mathbf{j}\mathbf{i_1} &=& -\mathbf{i_2}, \\ \mathbf{i_2}\mathbf{j} &=& \mathbf{j}\mathbf{i_2} &=& -\mathbf{i_1}. \end{array} \tag{2}$$

- Note that we define $\mathbb{C}(\mathbf{i}_k) := \{x + y\mathbf{i}_k \mid \mathbf{i}_k^2 = -1 \text{ and } x, y \in \mathbb{R}\}$ for $k = 1, 2$.

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In fact, the bicomplex numbers

$$\mathbb{T} \cong \mathrm{Cl}_{\mathbb{C}}(1,0) \cong \mathrm{Cl}_{\mathbb{C}}(0,1)$$

are *unique* among the complex Clifford algebras in that they are commutative but not division algebras. It is also convenient to write the set of bicomplex numbers as

$$\mathbb{T} := \{w_0 + w_1\mathbf{i_1} + w_2\mathbf{i_2} + w_3\mathbf{j} \mid w_0, w_1, w_2, w_3 \in \mathbb{R}\}. \tag{3}$$

- In particular, if we put $z_1 = x$ and $z_2 = y\mathbf{i_1}$ with $x, y \in \mathbb{R}$ in $z_1 + z_2\mathbf{i_2}$, then we obtain the following subalgebra of hyperbolic numbers, also called duplex numbers:

$$\mathbb{D} := \{x + y\mathbf{j} \mid \mathbf{j}^2 = 1,\ x, y \in \mathbb{R}\} \cong \mathrm{Cl}_{\mathbb{R}}(0,1).$$

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- Complex conjugation plays an important role both for algebraic and geometric properties of $\mathbb{C}$. For bicomplex numbers, there are three possible conjugations. Let $w \in \mathbb{T}$ and $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ such that $w = z_1 + z_2\mathbf{i_2}$. Then we define the three conjugations as:

$$w^{\dagger_1} = (z_1 + z_2\mathbf{i_2})^{\dagger_1} := \overline{z}_1 + \overline{z}_2\mathbf{i_2}, \quad (4a)$$

$$w^{\dagger_2} = (z_1 + z_2\mathbf{i_2})^{\dagger_2} := z_1 - z_2\mathbf{i_2}, \quad (4b)$$

$$w^{\dagger_3} = (z_1 + z_2\mathbf{i_2})^{\dagger_3} := \overline{z}_1 - \overline{z}_2\mathbf{i_2}, \quad (4c)$$

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We know that the product of a standard complex number with its conjugate gives the square of the Euclidean metric in $\mathbb{R}^2$. The analogs of this, for bicomplex numbers, are the following. Let $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ and $w = z_1 + z_2\mathbf{i_2} \in \mathbb{T}$, then we have that:

$$|w|^2_{\mathbf{i_1}} := w \cdot w^{\dagger_2} \in \mathbb{C}(\mathbf{i_1}), \tag{5a}$$

$$|w|^2_{\mathbf{i_2}} := w \cdot w^{\dagger_1} \in \mathbb{C}(\mathbf{i_2}), \tag{5b}$$

$$|w|^2_{\mathbf{j}} := w \cdot w^{\dagger_3} \in \mathbb{D}. \tag{5c}$$

- It is also important to know that every bicomplex number $z_1 + z_2\mathbf{i_2}$ has the following unique idempotent representation:

$$z_1 + z_2\mathbf{i_2} = (z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2}. \tag{6}$$

where $\mathbf{e_1} = \frac{1+\mathbf{j}}{2}$ and $\mathbf{e_2} = \frac{1-\mathbf{j}}{2}$.

- This representation is very useful because: addition, multiplication and division can be done term-by-term. Also, an element will be non-invertible if and only if $z_1 - z_2\mathbf{i_1} = 0$ or $z_1 + z_2\mathbf{i_1} = 0$.
- We say that $X \subseteq \mathbb{T}$ is a $\mathbb{T}$-cartesian set determined by X_1 and X_2 if $X = X_1 \times_e X_2$ where

$$X_1 \times_e X_2 := \{z_1 + z_2\mathbf{i_2} \in \mathbb{T} : z_1 + z_2\mathbf{i_2} = w_1\mathbf{e_1} + w_2\mathbf{e_2}, (w_1, w_2) \in X_1 \times X_2\}.$$

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Let U be an open set of $\mathbb{T}$ and $w_0 \in U$. Then, $f : U \subseteq \mathbb{T} \longrightarrow \mathbb{T}$ is said to be $\mathbb{T}$-differentiable at w_0 with derivative equal to $f'(w_0) \in \mathbb{T}$ if

$$\lim_{\substack{w \longrightarrow w_0 \\ (w - w_0 \ inv.)}} \frac{f(w) - f(w_0)}{w - w_0} = f'(w_0).$$

- We also say that the function f is $\mathbb{T}$-holomorphic on an open set U if and only if f is $\mathbb{T}$-differentiable at each point of U.

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- As we saw, a bicomplex number can be seen as an element of $\mathbb{C}^2$, so a function $f(z_1 + z_2\mathbf{i_2}) = f_1(z_1, z_2) + f_2(z_1, z_2)\mathbf{i_2}$ of $\mathbb{T}$ can be seen as a mapping $f(z_1, z_2) = (f_1(z_1, z_2), f_2(z_1, z_2))$ of $\mathbb{C}^2$. Here we have a characterization of such mappings:

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Let U be an open set and $f : U \subseteq \mathbb{T} \longrightarrow \mathbb{T}$. Let also $f(z_1 + z_2\mathbf{i_2}) = f_1(z_1, z_2) + f_2(z_1, z_2)\mathbf{i_2}$. Then f is $\mathbb{T}$-holomorphic on U if and only if:

$$f_1 \text{ and } f_2 \text{ are holomorphic in } z_1 \text{ and } z_2$$

and,

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \text{ and } \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \text{ on U.}$$

Moreover, $f' = \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_1}\mathbf{i_2}$ and $f'(w)$ is invertible if and only if $det\mathcal{J}_f(w) \neq 0$.

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- Hence, it is natural to define the corresponding class of mappings for $\mathbb{C}^2$:

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The class of $\mathbb{T}$-holomorphic mappings on a open set $U \subseteq \mathbb{C}^2$ is defined as follows:

$$TH(U) := \{f : U \subseteq \mathbb{C}^2 \longrightarrow \mathbb{C}^2 | f \in \mathrm{H}(U) \text{ and } \frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2}\}.$$

It is the subclass of holomorphic mappings of $\mathbb{C}^2$ satisfying the complexified Cauchy-Riemann equations.

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- We will first consider the variable $z = x + y\mathbf{i_1}$, where x and y are real variables and the corresponding formal differential operators

$$\partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + \mathbf{i_1}\partial_y\right) \text{ and } \partial_z = \frac{1}{2}\left(\partial_x - \mathbf{i_1}\partial_y\right).$$

Notation $f_{\bar{z}}$ or f_z means the application of $\partial_{\bar{z}}$ or ∂_z respectively to a bicomplex function $f(z) = u(z) + v(z)\mathbf{i_1} + r(z)\mathbf{i_2} + s(z)\mathbf{j}$. The derivatives f_z, $f_{\bar{z}}$ "exist" if and only if f_x and f_y do.

In view of these operators,

$$f_{\bar{z}}(z) = 0 \Leftrightarrow \partial_{\bar{z}}[u(z) + v(z)\mathbf{i_1}] = 0 \text{ and } \partial_{\bar{z}}[r(z) + s(z)\mathbf{i_1}] = 0. \qquad (7)$$

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Let $f(z_1+z_2\mathbf{i_2})=f_1(z_1,z_2)+f_2(z_1,z_2)\mathbf{i_2}=u(z_1,z_2)+v(z_1,z_2)\mathbf{i_1}+r(z_1,z_2)\mathbf{i_2}+s(z_1,z_2)\mathbf{j}$ be a bicomplex function. If the derivative

$$f'(\omega_0)=\lim_{\substack{\omega\to\omega_0\\(\omega-\omega_0\ \ inv.)}}\frac{f(\omega)-f(\omega_0)}{\omega-\omega_0} \tag{8}$$

exists, then $u_x, u_y, r_x, r_y, v_x, v_y, s_x$ and s_y exist, and

1. $f_\omega(\omega_0)=f'(\omega_0)$ (9)
2. $f_{\omega^{\dagger_1}}(\omega_0)=0$ (10)
3. $f_{\omega^{\dagger_2}}(\omega_0)=0$ (11)
4. $f_{\omega^{\dagger_3}}(\omega_0)=0.$ (12)

Moreover, if $u_x, u_y, v_x, v_y, r_x, r_y, s_x$ and s_y exist, and are continuous in a neighborhood of ω_0, and if (10), (11) and (12) hold, then (9) exists.

Class R1

Our bicomplex generalization of function theory is based on the following three different representations of bicomplex numbers. The *scalar* and *vectorial* part must be adapted to each representations.

Definition [Class R1]

Let $a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j} = z_1 + z_2\mathbf{i_2}$ where $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$. In this case, the theory will be based on assigning the part played by 1 and $\mathbf{i_2}$ to two essentially arbitrary bicomplex functions $F(\omega)$ and $G(\omega)$. We assume that these functions are defined and twice continuously differentiable in some open domain $D_0 \subset \mathbb{T}$. We require that

$$\mathrm{Vec}\{F(\omega)^{\dagger_2} G(\omega)\} \neq 0. \tag{13}$$

Under this condition, (F, G) will be called a $\mathbf{i_1}$-generating pair in D_0.

- It that case, for every ω_0 in D_0 we can find **unique** constants $\lambda_0, \mu_0 \in \mathbb{C}(\mathbf{i_1})$ such that $w(\omega_0) = \lambda_0 F(\omega_0) + \mu_0 G(\omega_0)$.

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Now, we say that $w(\omega) : D_0 \subset \mathbb{T} \rightarrow \mathbb{T}$ possesses at ω_0 the $(F, G)_{\mathbf{i_1}}$-derivative $\dot{w}(\omega_0)$ if the (finite) limit

$$\dot{w}(\omega_0) = \lim_{\substack{\omega \rightarrow \omega_0 \\ (\omega - \omega_0 \ inv.)}} \frac{w(\omega) - \lambda_0 F(\omega) - \mu_0 G(\omega)}{\omega - \omega_0} \tag{14}$$

exists.

- In the particular case where $w(\omega), F(\omega)$ and $G(\omega)$ are defined on $D_0 \subset \mathbb{C}(\mathbf{i_2}) \rightarrow \mathbb{C}(\mathbf{i_2})$ then we can find unique constants $\lambda_0, \mu_0 \in \mathbb{R}$ such that $w(\omega_0) = \lambda_0 F(\omega_0) + \mu_0 G(\omega_0)$ and we come back to the classical (in $\mathbf{i_2}$) pseudoanalytic developed by L. Bers and I.N. Vekua. In that case, using Bers's theory of Taylor series for pseudoanalytic function, V.V. Kravchenko obtain a locally complete system of solutions of the real stationary two-dimensional Schrödinger equation.

Class R1

Definition

Now, we say that $w(\omega) : D_0 \subset \mathbb{T} \to \mathbb{T}$ possesses at ω_0 the $(F, G)_{\mathbf{i_1}}$-derivative $\dot{w}(\omega_0)$ if the (finite) limit

$$\dot{w}(\omega_0) = \lim_{\substack{\omega \to \omega_0 \\ (\omega - \omega_0 \ inv.)}} \frac{w(\omega) - \lambda_0 F(\omega) - \mu_0 G(\omega)}{\omega - \omega_0} \qquad (14)$$

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Class R1

- On the other hand, in the case where $w(\omega), F(\omega)$ and $G(\omega)$ are defined on $D_0 \subset \mathbb{C}(\mathbf{j}) \to \mathbb{C}(\mathbf{j})$ then we can also find unique constants $\lambda_0, \mu_0 \in \mathbb{R}$ such that $w(\omega_0) = \lambda_0 F(\omega_0) + \mu_0 G(\omega_0)$ and we are in the hyperbolic pseudoanalytic theory developed by Guo Chun Wen.
- Moreover, if we only restrict the domain D_0 to $\mathbb{C}(\mathbf{i_2})$, the subclass of bicomplex pseudoanalytic functions obtained is precisely the class developed by V.V. Kravchenko and A. Castañeda to show that in a two-dimensional situation the Dirac equation with a scalar and an electromagnetic potentials decouples into a pair of bicomplex equations. It is also the same class of functions that used V.V. Kravchenko to obtain solutions of the complex stationary two-dimensional Schrödinger equation. However, the case using the complex functions is more complicated and the proof of expansion and convergence theorems for that type of bicomplex Vekua equation is still open.

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Definition

The following expressions are called the $\mathbf{i_1}$-characteristic coefficients of the pair (F, G) for $k = 1, 2, 3$:

$$a_{(F,G)}^{(k)} = -\frac{F^{\dagger_k} G_{\omega^{\dagger_k}} - F_{\omega^{\dagger_k}} G^{\dagger_k}}{FG^{\dagger_2} - F^{\dagger_2} G}, \qquad b_{(F,G)}^{(k)} = \frac{FG_{\omega^{\dagger_k}} - F_{\omega^{\dagger_k}} G}{FG^{\dagger_2} - F^{\dagger_2} G},$$

$$A_{(F,G)} = -\frac{F^{\dagger_2} G_{\omega} - F_{\omega} G^{\dagger_2}}{FG^{\dagger_2} - F^{\dagger_2} G}, \qquad B_{(F,G)} = \frac{FG_{\omega} - F_{\omega} G}{FG^{\dagger_2} - F^{\dagger_2} G}.$$

Class R1

Theorem

Let (F,G) be a $\mathbf{i_1}$-generating pair in some open domain D_0. Every bicomplex function w defined in D_0 admits the unique representation $w = \phi F + \psi G$ where $\phi, \psi : D_0 \subset \mathbb{T} \to \mathbb{C}(\mathbf{i_1})$. Moreover, the $(F,G)_{\mathbf{i_1}}$-derivative $\dot{w} = \frac{d_{(F,G)_{\mathbf{i_1}}} w}{d\omega}$ of $w(\omega)$ exists at ω_0 and has the form

$$\dot{w} = \phi_\omega F + \psi_\omega G = w_\omega - A_{(F,G)} w - B_{(F,G)} w^{\dagger_2} \tag{15}$$

if and only if

$$w_{\omega^{\dagger_1}} = a^{(1)}_{(F,G)} w + b^{(1)}_{(F,G)} w^{\dagger_2}, \tag{16}$$

$$w_{\omega^{\dagger_2}} = a^{(2)}_{(F,G)} w + b^{(2)}_{(F,G)} w^{\dagger_2}, \tag{17}$$

and

$$w_{\omega^{\dagger_3}} = a^{(3)}_{(F,G)} w + b^{(3)}_{(F,G)} w^{\dagger_2} \tag{18}$$

where w has continuous partial derivatives in a neighborhood of ω_0.

Class R1

Definition

The equations (16), (17) and (18) are called the $\mathbf{i_1}$-bicomplex Vekua equations and the solutions of these equations will be the $(F, G)_{\mathbf{i_1}}$-pseudoanalytic functions.

Remark

For $k = 1, 2, 3$, the equation

$$w_{\omega^{\dagger_k}} = a^{(k)}_{(F,G)} w + b^{(k)}_{(F,G)} w^{\dagger_2} \quad (19)$$

is equivalent to $\phi_{\omega^{\dagger_k}} F + \psi_{\omega^{\dagger_k}} G = 0$ if and only if

$$[G^{\dagger_k} - G^{\dagger_2}] F_{\omega^{\dagger_k}} = [F^{\dagger_k} - F^{\dagger_2}] G_{\omega^{\dagger_k}}. \quad (20)$$

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Class R2

Definition [Class R2]

Let $a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j} = z_1 + z_2\mathbf{i_1}$ where $z_1, z_2 \in \mathbb{C}(\mathbf{i_2})$. In this case, the theory will be based on assigning the part played by 1 and $\mathbf{i_1}$ to two essentially arbitrary bicomplex functions $F(\omega)$ and $G(\omega)$. We assume that these functions are defined and twice continuously differentiable in some open domain $D_0 \subset \mathbb{T}$. We require that

$$\mathrm{Vec}\{F(\omega)^{\dagger_1} G(\omega)\} \neq 0. \tag{21}$$

Under this condition, (F, G) will be called a $\mathbf{i_2}$-generating pair in D_0.

- It that case, for every ω_0 in D_0 we can find **unique** constants $\lambda_0, \mu_0 \in \mathbb{C}(\mathbf{i_2})$ such that $w(\omega_0) = \lambda_0 F(\omega_0) + \mu_0 G(\omega_0)$.

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Definition

We say that $w(\omega) : D_0 \subset \mathbb{T} \to \mathbb{T}$ possesses at ω_0 the $(F, G)_{\mathbf{i_2}}$-derivative $\dot{w}(\omega_0)$ if the (finite) limit

$$\dot{w}(\omega_0) = \lim_{\substack{\omega \to \omega_0 \\ (\omega - \omega_0 \ inv.)}} \frac{w(\omega) - \lambda_0 F(\omega) - \mu_0 G(\omega)}{\omega - \omega_0} \tag{22}$$

exists.

- In fact, if we interchange everywhere $\mathbf{i_1}$ with $\mathbf{i_2}$, this case is exactly the same than **R1**.

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Class R2

In particular, if we defined the function $\pi : \mathbb{T} \longrightarrow \mathbb{T}$ as

$$\pi(a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j}) := a + c\mathbf{i_1} + b\mathbf{i_2} + d\mathbf{j} \tag{23}$$

we obtain that $w(\omega)$ possesses a $(F, G)_{\mathbf{i_1}}$-derivative at $\omega_0 \in D_0$ if and only if the function

$$(\pi \circ w \circ \pi)(\omega) \tag{24}$$

possesses a $(\pi \circ F \circ \pi, \pi \circ G \circ \pi)_{\mathbf{i_2}}$-derivative at $\pi(\omega_0) \in \pi(D_0)$ where

$$(\pi \circ F \circ \pi, \pi \circ G \circ \pi) \tag{25}$$

is a $\mathbf{i_2}$-generating pair on $\pi(D_0)$.

Class R3

Definition [Class R3]

Let $a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j} = z_1 + z_2\mathbf{i_1}$ (resp. $z_1 + z_3\mathbf{i_2}$) where $z_1, z_2, z_3 \in \mathbb{C}(\mathbf{j})$. In this case, the theory will be based on assigning the part played by 1 and $\mathbf{i_1}$ (resp. $\mathbf{i_2}$) to two essentially arbitrary bicomplex functions $F(z)$ and $G(z)$. We assume that these functions are defined and twice continuously differentiable in some open domain $D_0 \subset \mathbb{T}$. We require that

$$\mathrm{Vec}\{F(\omega)^{\dagger_3} G(\omega)\} \neq 0. \tag{26}$$

Under this condition, (F, G) will be called a $\mathbf{j}$-generating pair in D_0.

- It that case, for every ω_0 in D_0 we can find **unique** constants $\lambda_0, \mu_0 \in \mathbb{C}(\mathbf{j})$ such that $w(\omega_0) = \lambda_0 F(\omega_0) + \mu_0 G(\omega_0)$.

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$$\dot{w}(\omega_0) = \lim_{\substack{\omega \to \omega_0 \\ (\omega - \omega_0 \ inv.)}} \frac{w(\omega) - \lambda_0 F(\omega) - \mu_0 G(\omega)}{\omega - \omega_0} \tag{27}$$

exists.

- In this case, the following expressions are called the $\mathbf{j}$-characteristic coefficients of the pair (F, G) for $k = 1, 2, 3$:

$$a_{(F,G)}^{(k)} = -\frac{F^{\dagger_k} G_{\omega^{\dagger_k}} - F_{\omega^{\dagger_k}} G^{\dagger_k}}{FG^{\dagger_3} - F^{\dagger_3} G}, \qquad b_{(F,G)}^{(k)} = \frac{FG_{\omega^{\dagger_k}} - F_{\omega^{\dagger_k}} G}{FG^{\dagger_3} - F^{\dagger_3} G},$$

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Class R3

Theorem

Let (F,G) be a **j**-generating pair in some open domain D_0. Every bicomplex function w defined in D_0 admits the unique representation $w = \phi F + \psi G$ where $\phi, \psi : D_0 \subset \mathbb{T} \to \mathbb{C}(\mathbf{j})$. Moreover, the $(F, G)_{\mathbf{j}}$-derivative $\dot{w} = \frac{d_{(F,G)_{\mathbf{j}}} w}{d\omega}$ of $w(\omega)$ exists at ω_0 and has the form

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if and only if

$$w_{\omega^{\dagger_1}} = a^{(1)}_{(F,G)} w + b^{(1)}_{(F,G)} w^{\dagger_3}, \quad (29)$$

$$w_{\omega^{\dagger_2}} = a^{(2)}_{(F,G)} w + b^{(2)}_{(F,G)} w^{\dagger_3}, \quad (30)$$

and

$$w_{\omega^{\dagger_3}} = a^{(3)}_{(F,G)} w + b^{(3)}_{(F,G)} w^{\dagger_3} \quad (31)$$

where w has continuous partial derivatives in a neighborhood of ω_0.

Class R3

- In this case, it is useful to consider a more specific class of generating pair.

Definition

Let D_1 and D_2 be open in $\mathbb{C}(\mathbf{i_1})$. Consider that (F_{e_1}, G_{e_1}) and (F_{e_2}, G_{e_2}) are complex (in $\mathbf{i_1}$), twice continuously differentiable, generating pairs in respectively D_1 and D_2. Under these conditions, (F, G) will be called a $\mathbf{j}^*$-generating pair in $D_0 = D_1 \times_e D_2 \in \mathbb{T}$ where

$$F(z_1 + z_2\mathbf{i_2}) := F_{e_1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + F_{e_2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2} \qquad (32)$$

and

$$G(z_1 + z_2\mathbf{i_2}) := G_{e_1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + G_{e_2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}. \qquad (33)$$

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Class R3

Lemma

Let $F(\omega)$ and $G(\omega)$ two arbitrary bicomplex functions defined in some domain $D_0 \subset \mathbb{T}$. If

$$\mathrm{Im}\{\overline{F_{e_1}(\omega)}G_{e_1}(\omega)\} \neq 0 \text{ or } \mathrm{Im}\{\overline{F_{e_2}(\omega)}G_{e_2}(\omega)\} \neq 0 \ \forall \omega \in D_0$$

then $\mathrm{Vec}\{F(\omega)^{\dagger_3}G(\omega)\} \neq 0 \ \forall \omega \in D_0$.

Therefore, from the last lemma, we obtain automatically this following result.

Theorem

Let $D_0 = D_1 \times_e D_2$ where D_1 and D_2 are open domains in $\mathbb{C}(\mathbf{i_1})$. If (F, G) is a $\mathbf{j}^*$-generating pair in D_0 then (F, G) is, in particular, a $\mathbf{j}$-generating pair in D_0.

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Class R3

The following result established an explicit connection between the **j**-bicomplex Vekua equations of two complex variables and the classical Vekua equations.

Theorem

If (F_{e_1}, G_{e_1}) and (F_{e_2}, G_{e_2}) are complex (in $\mathbf{i_1}$) generating pairs in respectively D_1 and D_2. Then w is a solution on $D_0 = D_1 \times_e D_2$ of the **j**-bicomplex Vekua equations with the $\mathbf{j}^*$-generating pair (F, G) if and only if $w(z_1 + z_2\mathbf{i_2}) = w_{e1}(z_1 - z_2\mathbf{i_1})\mathbf{e_1} + w_{e2}(z_1 + z_2\mathbf{i_1})\mathbf{e_2}$ where w_{e_k} is a solution on D_k of the complex (in $\mathbf{i_1}$) Vekua equation with the generating pair (F_{e_k}, G_{e_k}) for $k = 1, 2$.

The Complexified Schrödinger Equation

Definition

Consider the equation

$$(\triangle_{\mathbb{C}} - \nu(z_1, z_2))f = 0 \tag{34}$$

in $\Omega \subset \mathbb{R}^4$, where $\triangle_{\mathbb{C}} = \partial^2_{z_1} + \partial^2_{z_2}$, ν and f are complex (in $\mathbf{i_1}$) valued functions. The equation (34) is simply the complexification of the two-dimensional stationary Schrödinger equation where $\triangle_{\mathbb{C}}$ is the **complex Laplacian**.

The Complex Laplacian

First of all, we will write the complex Laplacian in a more explicit way.

Lemma

Let $\omega = z_1 + z_2\mathbf{i_2}$, where $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ then

$$\partial_\omega \partial_{\bar{\omega}} = \frac{1}{4}(\partial^2_{z_1} + \partial^2_{z_2}) = \frac{1}{4}\triangle_{\mathbb{C}}$$

$\forall f \in C^2(\Omega)$ where Ω is an open set in $\mathbb{R}^4$.

Proposition

Let $\partial_{z_1} = \frac{1}{2}\left(\partial_x - \mathbf{i_1}\partial_y\right)$ and $\partial_{z_2} = \frac{1}{2}\left(\partial_p - \mathbf{i_1}\partial_q\right)$ then

$$16\partial_\omega\partial_{\bar{\omega}} = 4\triangle_{\mathbb{C}} = \left(\partial^2_x - \partial^2_y + \partial^2_p - \partial^2_q\right) - 2\mathbf{i_1}\left(\partial^2_{xy} + \partial^2_{pq}\right)$$

$\forall f \in C^2(\Omega)$ where Ω is an open set in $\mathbb{R}^4$.

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$\forall f \in C^2(\Omega)$ where Ω is an open set in $\mathbb{R}^4$.

The Complex Laplacian

Remark

In the last proposition, if we let y and q be constant variables, then

1. $\Omega \subset \mathbb{C}(\mathbf{i_2})$;
2. $4\partial_\omega \partial_{\bar{\omega}} = \partial_z \partial_{\bar{z}}$ where $\partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + \mathbf{i_2}\partial_p\right)$ and $\partial_z = \frac{1}{2}\left(\partial_x - \mathbf{i_2}\partial_p\right)$;
3. $4\triangle_{\mathbb{C}} = 4\partial_z \partial_{\bar{z}} = \partial_x^2 + \partial_p^2 = \triangle$, **the Laplacian operator**.

Similarly, if y and p are constant variables, then

1. $\Omega \subset \mathbb{D}$;
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In the last proposition, if we let y and q be constant variables, then

1. $\Omega \subset \mathbb{C}(\mathbf{i_2})$;
2. $4\partial_\omega\partial_{\bar{\omega}} = \partial_z\partial_{\bar{z}}$ where $\partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + \mathbf{i_2}\partial_p\right)$ and $\partial_z = \frac{1}{2}\left(\partial_x - \mathbf{i_2}\partial_p\right)$;
3. $4\triangle_{\mathbb{C}} = 4\partial_z\partial_{\bar{z}} = \partial_x^2 + \partial_p^2 = \triangle$, **the Laplacian operator**.

Similarly, if y and p are constant variables, then

1. $\Omega \subset \mathbb{D}$;
2. $4\partial_\omega\partial_{\bar{\omega}} = \partial_z\partial_{\bar{z}}$ where $\partial_{\bar{z}} = \frac{1}{2}\left(\partial_x - \mathbf{j}\partial_q\right)$ and $\partial_z = \frac{1}{2}\left(\partial_x + \mathbf{j}\partial_q\right)$;
3. $4\triangle_{\mathbb{C}} = 4\partial_z\partial_{\bar{z}} = \partial_x^2 - \partial_q^2 = \Box$, **the wave operator**.

Factorization of the Complexified Schrödinger Operator

It is well known that if f_0 is a nonvanishing particular solution of the one-dimensional stationary schrödinger equation

$$\left(-\frac{d^2}{dx^2}+\nu(x)\right)$$

then the Scrödinger operator can be factorized as follows:

$$-\frac{d^2}{dx^2}+\nu(x)=\left(\frac{d}{dx}+\frac{f_0'}{f_0}\right)\left(\frac{d}{dx}-\frac{f_0'}{f_0}\right).$$

By C we denote the $\dagger_2$-bicomplex conjugation operator.

Theorem

Let $f_0:\Omega\subset\mathbb{R}^4\longrightarrow\mathbb{C}(\mathbf{i_1})$ be a nonvanishing particular solution of (34). Then for any $\mathbb{C}(\mathbf{i_1})$-valued continuously twice differentiable function φ the following equality hold:

$$(\triangle_{\mathbb{C}}-\nu)\varphi=4\left(\partial_{\bar{\omega}}+\frac{\partial_\omega f_0}{f_0}C\right)\left(\partial_\omega-\frac{\partial_\omega f_0}{f_0}C\right)\varphi. \qquad (35)$$

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Factorization of the Complexified Schrödinger Operator

Remark

From the last Remark, we see that the complexified Schrödinger equation contains the stationary two-dimensional Schrödinger equation

$$(\triangle - \nu(x, p))f = 0$$

and the Klein-Gordon equation

$$(\square - \nu(x, q))f = 0.$$

Hence, our factorization of the complexified Schrödinger equation is a generalization of the factorization already obtained for the stationary two-dimensional Schrödinger equation and for the Klein-Gordon equation.

Solutions of the Complexified Schrödinger Equation

Theorem

Let W be a solution of the following bicomplex Vekua equation

$$\left(\partial_{\omega^{\dagger_2}} - \frac{\partial_{\omega^{\dagger_2}} f_0}{f_0} C\right) W = 0 \tag{36}$$

where f_0 is a nonvanishing solution of the complexified Schrödinger equation (34). Then $u = \mathrm{Sc}(W)$ is a solution of (34) and $v = \mathrm{Vec}(W)$ is a solution of the equation

$$\left(\triangle_{\mathbb{C}} + \nu(z_1, z_2) - 2\left(\frac{|\nabla_{\mathbb{C}} f_0|_{\mathbf{i_1}}}{f_0}\right)^2\right) v = 0 \tag{37}$$

where $\nabla_{\mathbb{C}} = \partial_{z_1} + \mathbf{i_2}\partial_{z_2}$.

Solutions of the Complexified Schrödinger Equation

Remark

If W possesses a $(f_0, \frac{\mathbf{i_2}}{f_0})_{\mathbf{i_1}}$-derivative on an open set $\Omega \subset \mathbb{T}$ then W is a solution of the bicomplex Vekua equation:

$$\left(\partial_{\omega^{\dagger_2}} - \frac{\partial_{\omega^{\dagger_2}} f_0}{f_0} C\right) W = 0 \text{ on } \Omega.$$

In that case, $a_{(F,G)}^{(2)} = 0$ and $b_{(F,G)}^{(2)} = \frac{\partial_{\omega^{\dagger_2}} f_0}{f_0}$ where

$$F = f_0 \quad \text{and} \quad G = \frac{\mathbf{i_2}}{f_0}$$

is a $\mathbf{i_1}$-generating pair for (36). Hence, the bicomplex pseudoanalytic function theory open the way to find explicit solutions of the complexified Schrödinger equation.