Normal Families of Bicomplex Holomorphic Functions

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Definition

Bicomplex numbers are defined as

$$\mathbb{T} := \{z_1 + z_2\mathbf{i_2} \mid z_1, z_2 \in \mathbb{C}(\mathbf{i_1})\} \tag{1}$$

where the imaginary units $\mathbf{i_1}, \mathbf{i_2}$ and $\mathbf{j}$ are governed by the rules:
$\mathbf{i_1}^2 = \mathbf{i_2}^2 = -1$, $\mathbf{j}^2 = 1$ and

$$\begin{array}{rcccr} \mathbf{i_1}\mathbf{i_2} & = & \mathbf{i_2}\mathbf{i_1} & = & \mathbf{j}, \\ \mathbf{i_1}\mathbf{j} & = & \mathbf{j}\mathbf{i_1} & = & -\mathbf{i_2}, \\ \mathbf{i_2}\mathbf{j} & = & \mathbf{j}\mathbf{i_2} & = & -\mathbf{i_1}. \end{array} \tag{2}$$

- Note that we define $\mathbb{C}(\mathbf{i}_k) := \{x + y\mathbf{i}_k \mid \mathbf{i}_k^2 = -1 \text{ and } x, y \in \mathbb{R}\}$ for $k = 1, 2$.

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In fact, bicomplex numbers

$$\mathbb{T} \cong \mathrm{Cl}_{\mathbb{C}}(1,0) \cong \mathrm{Cl}_{\mathbb{C}}(0,1)$$

are *unique* among the complex Clifford algebras in that they are commutative but not division algebra. It is also convenient to write the set of bicomplex numbers as

$$\mathbb{T} := \{w_0 + w_1\mathbf{i_1} + w_2\mathbf{i_2} + w_3\mathbf{j} \mid w_0, w_1, w_2, w_3 \in \mathbb{R}\}. \quad (3)$$

- In particular, if we put $z_1 = x$ and $z_2 = y\mathbf{i_1}$ with $x, y \in \mathbb{R}$ in $z_1 + z_2\mathbf{i_2}$, then we obtain the following subalgebra of hyperbolic numbers, also called duplex numbers:

$$\mathbb{D} := \{x + y\mathbf{j} \mid \mathbf{j}^2 = 1,\ x, y \in \mathbb{R}\} \cong \mathrm{Cl}_{\mathbb{R}}(0,1).$$

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- It is also important to know that every bicomplex number $z_1 + z_2\mathbf{i_2}$ has the following unique idempotent representation:

$$z_1 + z_2\mathbf{i_2} = (z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2}. \tag{4}$$

where $\mathbf{e_1} = \frac{1+\mathbf{j}}{2}$ and $\mathbf{e_2} = \frac{1-\mathbf{j}}{2}$.

- We define the projections $\mathcal{P}_1, \mathcal{P}_2 : \mathbb{T} \longrightarrow \mathbb{C}(\mathbf{i_1})$ as

$$\mathcal{P}_1(z_1 + z_2\mathbf{i_2}) = z_1 - z_2\mathbf{i_1} \text{ and } \mathcal{P}_2(z_1 + z_2\mathbf{i_2}) = z_1 + z_2\mathbf{i_1}.$$

An element $w = z_1 + z_2\mathbf{i_2}$ will be non-invertible if and only if $\mathcal{P}_1(w) = 0$ or $\mathcal{P}_2(w) = 0$.

- We say that $X \subseteq \mathbb{T}$ is a $\mathbb{T}$-cartesian set determined by X_1 and X_2 if $X = X_1 \times_e X_2$ where

$$X_1 \times_e X_2 := \{z_1 + z_2\mathbf{i_2} \in \mathbb{T} : z_1 + z_2\mathbf{i_2} = w_1\mathbf{e_1} + w_2\mathbf{e_2}, (w_1, w_2) \in X_1 \times X_2\}.$$

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Let U be an open set of $\mathbb{T}$ and $w_0 \in U$. Then, $f : U \subseteq \mathbb{T} \longrightarrow \mathbb{T}$ is said to be $\mathbb{T}$-differentiable at w_0 with derivative equal to $f'(w_0) \in \mathbb{T}$ if

$$\lim_{\substack{w \longrightarrow w_0 \\ (w - w_0 \ \ inv.)}} \frac{f(w) - f(w_0)}{w - w_0} = f'(w_0).$$

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- As we saw, a bicomplex number can be seen as an element of $\mathbb{C}^2$, so a function $f(z_1 + z_2\mathbf{i_2}) = f_1(z_1, z_2) + f_2(z_1, z_2)\mathbf{i_2}$ of $\mathbb{T}$ can be seen as a mapping $f(z_1, z_2) = (f_1(z_1, z_2), f_2(z_1, z_2))$ of $\mathbb{C}^2$. Here we have a characterization of such mappings:

Theorem

Let U be an open set and $f : U \subseteq \mathbb{T} \longrightarrow \mathbb{T}$. Let also $f(z_1 + z_2\mathbf{i_2}) = f_1(z_1, z_2) + f_2(z_1, z_2)\mathbf{i_2}$. Then f is $\mathbb{T}$-holomorphic on U if and only if:

$$f_1 \text{ and } f_2 \text{ are holomorphic in } z_1 \text{ and } z_2$$

and,

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \text{ and } \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \text{ on U.}$$

Moreover, $f' = \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_1}\mathbf{i_2}$ and $f'(w)$ is invertible if and only if $det\mathcal{J}_f(w) \neq 0$.

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- Hence, it is natural to define the corresponding class of mappings for $\mathbb{C}^2$:

Definition

The class of $\mathbb{T}$-holomorphic mappings on a open set $U \subseteq \mathbb{C}^2$ is defined as follows:

$$TH(U) := \{f : U \subseteq \mathbb{C}^2 \longrightarrow \mathbb{C}^2 | f \in \mathrm{H}(U) \text{ and } \frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2}\}.$$

It is the subclass of holomorphic mappings of $\mathbb{C}^2$ satisfying the complexified Cauchy-Riemann equations.

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A family **F** of bicomplex holomorphic functions defined on a domain $D \subseteq \mathbb{T}$ is said to be **locally uniformally bounded** in D if for every compact set $K \subset D$ there exists a constant $M(K)$ such that

$$\|f(z)\| \leq M \quad \forall f \in \mathbf{F}, \text{ and } \forall z \in K.$$

Definition

A sequence $\{f_n\}$ of bicomplex holomorphic functions defined on a domain $D \subseteq \mathbb{T}$ is said to **converge uniformally on compact subsets** of D to a bicomplex function f if for every compact subset K of D and for every $\epsilon > 0$ there is a positive integer n_0 such that

$$\|f_n(w) - f(w)\| < \epsilon \quad \forall n \geq n_0, \quad and \quad \forall w \in K.$$

This type of convergence is also known as **local uniform convergence** or **normal convergence**.

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Definition

A family **F** of bicomplex functions (not necessarily holomorphic) defined on a domain $D \subseteq \mathbb{T}$ is said to be **equicontinuous** on D if for every $\epsilon > 0$ there is a $\delta > 0$ such that

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whenever $\|z - w\| < \delta$.

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Bicomplex Montel Theorem

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A family **F** of bicomplex holomorphic functions defined on a domain $D \subseteq \mathbb{T}$ is said to be **normal** in D if every sequence in **F** contains a subsequence which converges locally uniformly on D. **F** is said to be **normal at a point** $z \in D$ if it is normal in some neighbourhood of z in D.

Theorem (Montel)

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Remark

The converse of Bicomplex Montel Theorem is also true. Indeed, suppose that **F** is normal and not locally uniformly bounded in D. Then in some closed discus $\overline{D}(a; r_1, r_2) := \overline{B^1(a_1 - a_2\mathbf{i_1}, r_1)} \times_e \overline{B^1(a_1 + a_2\mathbf{i_1}, r_2)}$ in the domain D, for each $n \in \mathbb{N}$ there is a function $f_n \in \mathbf{F}$ and a point $w_n \in \overline{D}(a; r_1, r_2)$ such that $\|f_n(w_n)\| > n$. Since **F** is normal, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ converging uniformly on $\overline{D}(a; r_1, r_2)$ to a bicomplex (holomorphic) function f. That is, for some positive integer n_0, we have

$$\|f_{n_k}(w) - f(w)\| < 1, \quad \forall k \geq n_0, \text{ and } w \in \overline{D}(a; r_1, r_2).$$

Thus, if $M = max_{z\in\overline{D}(a;r_1,r_2)}\|f(w)\|$, then $\|f_{n_k}(w)\| \leq 1 + M, \ \ \forall w \in \overline{D}(a; r_1, r_2)$ and this is a contradiction.

Montel Theorem in $\mathbb{C}^n$

- In this section, we want to show that it is possible to see the **Bicomplex Montel Theorem** as a particular case of the following Montel theorem of several complex variables.

Theorem

Let $D \subset \mathbb{C}^n$ be an open set and $\mathbf{F} \subset \mathcal{O}(D, \mathbb{C}^n)$ be a family of holomorphic mapping. Then the following are equivalent:

1. The family $\mathbf{F}$ is locally uniformly bounded.
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Bicomplex Weierstrass Theorem

- We notice that a family **F** is relatively compact in $\mathcal{O}(D, \mathbb{C}^n)$ if and only if **F** is a normal family. Therefore, our assumption will be proven if we can show that $TH(D)$ is closed in $\mathcal{O}(D, \mathbb{C}^2)$ with the compact convergence topology. Hence, the desired result is a direct consequence of the following **Bicomplex Weierstrass Theorem**.

Theorem (Weierstrass)

Let $\{f_n\}$ be a sequence of bicomplex holomorphic functions on a domain D which converges uniformly on compact subsets of D to a function f. Then f is bicomplex holomorphic in D, and the sequence of derivatives $\{f_n^{(k)}\}$ converges uniformly on compact subsets to $f^{(k)}$, $k = 1, 2, 3....$

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- We start with a **uniformly bounded** family $\mathbf{F}$ of bicomplex holomorphic functions. In this case, we can verify directly the following result.

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A family $\mathbf{F}$ of bicomplex holomorphic functions defined on a domain D is uniformly bounded on D if and only if $\mathbf{F}_{ei} = \mathcal{P}_i(\mathbf{F})$ is uniformly bounded on $\mathcal{P}_i(D)$, $i = 1, 2$.

- A similar result is also true for the concept of **equicontinuity**.

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- If we consider now a **locally uniformly bounded** family **F** of bicomplex holomorphic functions, we can prove a similar result since a set $K = \mathcal{P}_1(K) \times_e \mathcal{P}_2(K)$ is compact if and only if $\mathcal{P}_i(K)$ is compact for $i = 1, 2$.

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- Now, from the Montel Theorem (and its converse), we obtain automatically the following result.

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A family **F** of bicomplex holomorphic functions defined on a domain D is normal on D if and only if $\mathbf{F}_{ei} = \mathcal{P}_i(\mathbf{F})$ is normal on $\mathcal{P}_i(D)$ for $i = 1, 2$.

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- What happen if D is not a bicomplex cartesian product? In the case of uniformly bounded family of bicomplex holomorphic functions it is easy to verify that the result is true for any domain. In the case of normality, we need to recall the following results from the bicomplex function theory.

Remark

A domain $D \subseteq \mathbb{T}$ is a domain of holomorphism for functions of a bicomplex variable if and only if D is a $\mathbb{T}$-cartesian set, and if D is not a domain of holomorphism then $D \subsetneq \mathcal{P}_1(D) \times_e \mathcal{P}_2(D)$, and there exists a holomorphic function which is a bicomplex holomorphic continuation of the given function from D to $\mathcal{P}_1(D) \times_e \mathcal{P}_2(D)$.

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- Using the last Remark, we are able to establish the following results.

Theorem

The family $\mathbf{F}$ of bicomplex holomorphic functions is normal on the arbitrary domain D if and only if $\mathbf{F}_{ei} = \mathcal{P}_i(\mathbf{F})$ is normal on $\mathcal{P}_i(D)$ for $i = 1, 2$.

Corollary

If the family $\mathbf{F}$ of bicomplex holomorphic functions is normal on the arbitrary domain D, then $\mathbf{F}$ is normal on the larger domain $\mathcal{P}_1(D) \times_e \mathcal{P}_2(D)$.

Corollary

A family of bicomplex holomorphic functions $\mathbf{F}$ is normal in a arbitrary domain D if and only if $\mathbf{F}$ is normal at each point of D.

- To carry further the study of normal families of bicomplex holomorphic functions particularly to consider the dynamics of bicomplex holomorphic functions, we propose the following more general definition of normality.

Definition

A family **F** of bicomplex holomorphic functions defined on a domain $D \subseteq \mathbb{T}$ is said to be **normal** in D if every sequence in **F** contains a subsequence which on compact subsets of D either converges uniformly to a limit function or converges uniformly to ∞. **F** is said to be **normal at a point** $z \in D$ if it is normal in some neighborhood of z in D.

Remark

We say that a sequence $\{w_n\}$ of bicomplex numbers converges to ∞ if and only if the norm $\{\|w_n\|\}$ congerges to ∞.

- We note that our proofs of the Bicomplex Montel Theorem works also in this situation. However, as for one complex variable, the converse of the Bicomplex Montel Theorem will not remain valid with this more complete definition of normality.

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- Here is the counterexample for the other side.

Example

Let X_1 and X_2 be domains in $\mathbb{C}(\mathbf{i_1})$ containing the origin. Let $D = (X_1 \times_e X_2) - \{0\}$. Then D is not a bicomplex cartesian domain because $D \neq \mathcal{P}_1(D) \times_e \mathcal{P}_2(D)$. Now the family

$$\mathbf{F} = \{nw : w = z_1 + z_2\mathbf{i_2}, \quad n \in \mathbb{N}\}$$

is normal in the domain D (by the proposed definition of normality as above) but $\mathbf{F}_{ei} = \mathcal{P}_i(\mathbf{F})$ is not normal in $\mathcal{P}_i(D)$, $i = 1, 2$ as it contains the origin.

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- Moreover, the next example shows that the converse the last Theorem is not true even if the domain D is a bicomplex cartesian product.

Example

Consider the family

$$\mathcal{F} = \{R^{\circ n}(z) \mid \; R(z) = z^2 \text{ and } n \in \mathbb{N}\}$$

on $\mathbb{C}(\mathbf{i_1})$. Then $\mathcal{F}$ is normal on $D_1 = \{z : |z| > 1\} \subset \mathbb{C}(\mathbf{i_1})$ where here the limit function is identically infinite, but not normal on $\mathbb{C}(\mathbf{i_1})$ since $\{|z| = 1\} \subset \mathbb{C}(\mathbf{i_1})$. However, the bicomplex family

$$\mathbf{F} := \mathbf{F}_{e1}\mathbf{e_1} + \mathbf{F}_{e2}\mathbf{e_2} = \{R^{\circ n}(w) \mid \; R(w) = w^2 \text{ and } n \in \mathbb{N}\}$$

where $\mathbf{F}_{e1} = \mathbf{F}_{e2} = \mathcal{F}$, is normal in the following bicomplex cartesian product:

$$D_1 \times_e \mathbb{C}(\mathbf{i_1})$$

since the limit function is identically infinite.

Fatou and Julia Sets for Polynomials

- Let us conclude this talk with the following general definition of **Fatou** and **Julia** sets for bicomplex polynomials.

Definition

Let $P(\zeta)$ be a bicomplex polynomials. We define the bicomplex Julia set for P as

$$\mathcal{J}_2(P) = \{\zeta \in \mathbb{T} \mid \{P^{\circ n}(\zeta)\} \text{ is not normal}\}$$

and the bicomplex Fatou (or stable) set as

$$\mathcal{F}_2(P) = \mathbb{T} - \mathcal{J}_2(P).$$

- Hence, about each point $\zeta \in \mathcal{F}_2(P)$, there is a neighborhood N_ζ in which $\{P^{\circ n}(\zeta)\}$ is a normal family. Therefore, $\mathcal{F}_2(P)$ is an open set, the connected components of which are the maximal domains of normality of $\{P^{\circ n}(\zeta)\}$, and $\mathcal{J}_2(P)$ is a closed set.

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- From the last Theorem, we obtain this following inclusion:

$$\mathcal{J}_2(P) \subset \left\{ z_1 + z_2\mathbf{i_2} \in \mathbb{T} \mid \{[\mathcal{P}_1(P)]^{\circ n}(z_1 - z_2\mathbf{i_1})\}\} \text{ or } \tag{5}$$

$$\{[\mathcal{P}_2(P)]^{\circ n}(z_1 + z_2\mathbf{i_1})\} \text{ is not normal } \right\} \tag{6}$$

$$= [\mathcal{J}_1(\mathcal{P}_1(P)) \times_e \mathbb{C}(\mathbf{i_1})] \cup [\mathbb{C}(\mathbf{i_1}) \times_e \mathcal{J}_1(\mathcal{P}_2(P))]. \tag{7}$$

- However, from the last Example, we know that (5) cannot be transformed into equality. In fact, to obtain a characterization of bicomplex Julia sets in terms of one variable dynamics we need to use the concept of filled-in Julia set. As for the complex case, the bicomplex **filled-in Julia** set $K_2(P)$ of a polynomial P is define as the set of all points ζ of dynamical space that have bounded orbit with respect to P, that is to say:

$$\mathcal{K}_2(P) = \{\zeta \in \mathbb{T} \mid \{P^{\circ n}(\zeta)\} \nrightarrow \infty \text{ as } n \to \infty\}. \tag{8}$$

We remark that $\mathcal{K}_2(P)$ is a closed set.

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We remark that $\mathcal{K}_2(P)$ is a closed set.

- As for the classical case, we need to consider polynomials of degree $d \geq 2$ to be able to see a bicomplex Julia set as the boundary of a bicomplex filled-in Julia set. In fact, to decompose $P(w)$ in terms of two complex polynomials of $d \geq 2$, we must also consider non-degenerate bicomplex polynomials of the form $P(w) = a_d w^d + a_{d-1} w^{d-1} + ... + a_0$ where $a_d \notin \mathcal{NC}$. Under these specifications, we have the following result.

Theorem

Let $P(\zeta)$ be a non-degenerate bicomplex polynomials of degree $d \geq 2$. Then,

$$\partial \mathcal{K}_2(P) = \mathcal{J}_2(P).$$

- Moreover, using the idempotent representation, it is easy to see that the bicomplex filled-in Julia set $\mathcal{K}_2(P)$ can be expressed in terms of two filled-in Julia sets in the plane. More specifically,

$$\mathcal{K}_2(P) = \mathcal{K}_1(\mathcal{P}_1(P)) \times_e \mathcal{K}_1(\mathcal{P}_2(P)).$$

- Hence, since $\partial[\mathcal{K}_1(\mathcal{P}_1(P)) \times_e \mathcal{K}_1(\mathcal{P}_2(P))] = [\partial\mathcal{K}_1(\mathcal{P}_1(P)) \times_e \mathcal{K}_1(\mathcal{P}_2(P))] \cup [\mathcal{K}_1(\mathcal{P}_1(P)) \times_e \partial\mathcal{K}_1(\mathcal{P}_2(P))]$, we have the following characterization of the bicomplex Julia set $\mathcal{J}_2(P)$ in terms of one complex variable dynamics.

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Let $P(\zeta)$ be a non-degenerate bicomplex polynomials of degree $d \geq 2$. Then,

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Conclusion

In the particular case of the bicomplex quadratic polynomial

$$P_c(\zeta) = \zeta^2 + c,$$

the definitions of Julia, Fatou and filled-in Julia set of this article coincide with the definitions introduced by D. Rochon in 2000. Moreover, using some distance estimation formulas that can be used to ray traced slices of bicomplex filled-in Julia sets in dimension three, we obtain the following visual examples (see Fig. 1, 2 and 3) of some bicomplex Julia sets $\mathcal{K}_2(P_c)$ with the specific slice $\mathbf{j} = 0$.

Figure 1: $c = (0.27)\mathbf{e_1} + (0.27)\mathbf{e_2}$

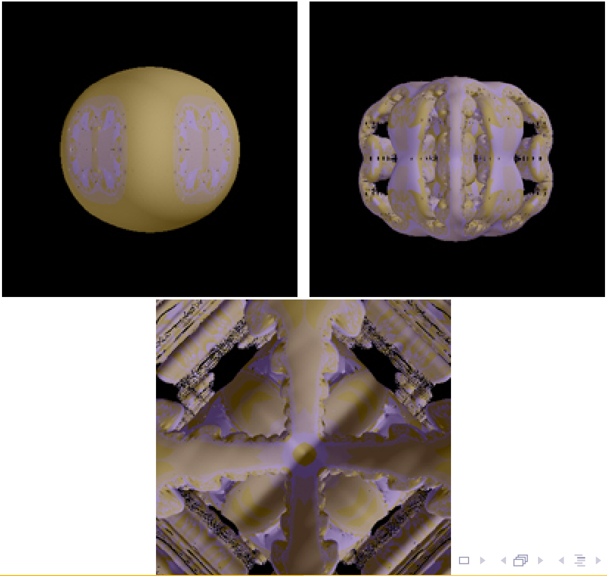


Figure 2: $c = (-1.754878)\mathbf{e_1} + (-1.754878)\mathbf{e_2}$

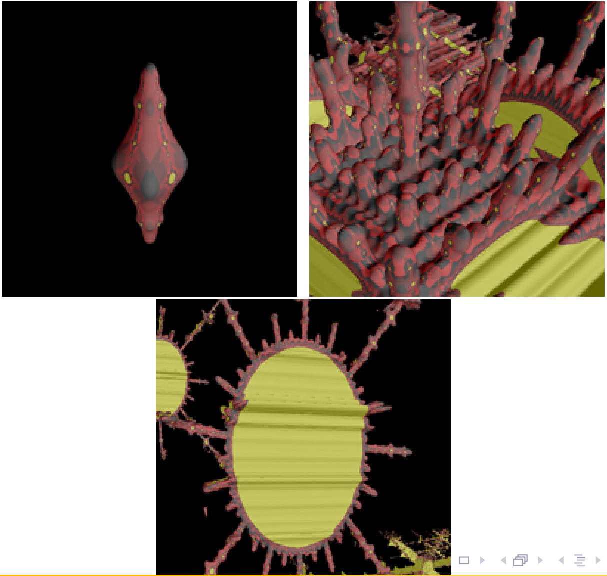
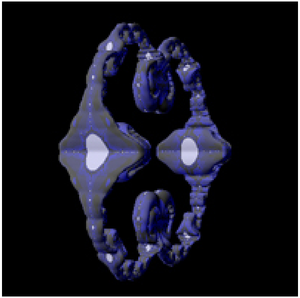
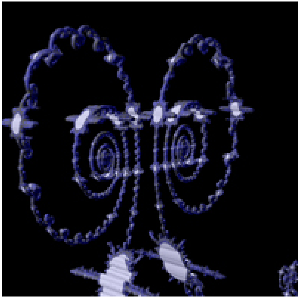
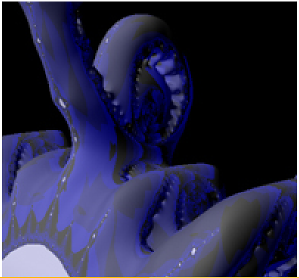


Figure 3: $c = (0.26)\mathbf{e_1} + (-1.754878)\mathbf{e_2}$







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