The Bicomplex Quantum Hydrogen-like Formalism

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Abstract

We obtain a complete analytical solution of the quantum-mechanical Coulomb potential problem formulated in terms of bicomplex numbers. We do so by solving the bicomplex three-dimensionnal eigenvalue equation associated with a hydrogen-like hamiltonian and obtaining explicit expressions for its eigenvalues and eigenfunctions. The same eigenvalues are obtained through Pauli's algebraic approach.

Introduction

Various generalizations of the complex number system that underlies the mathematical formulation of quantum mechanics have been known for some time, but the use of the commutative ring of bicomplex numbers for this purpose is a relatively new idea.

Bicomplex numbers and functions

One way to define a bicomplex number α is by writing

$$\alpha := \alpha_{\widehat{1}}\mathbf{e_1} + \alpha_{\widehat{2}}\mathbf{e_2}, \qquad \text{with} \qquad \alpha_{\widehat{1}}, \alpha_{\widehat{2}} \in \mathbb{C}(\mathbf{i_1}).$$

The imaginary units $\mathbf{e_1}$ and $\mathbf{e_2}$ satisfy the remarkable properties

$$\mathbf{e_1}^2 = \mathbf{e_1}, \quad \mathbf{e_2}^2 = \mathbf{e_2}, \quad \mathbf{e_1} + \mathbf{e_2} = 1 \quad \text{and} \quad \mathbf{e_1}\mathbf{e_2} = 0 = \mathbf{e_2}\mathbf{e_1}.$$

With the addition and multiplication defined in the obvious way, the set of bicomplex numbers $\mathbb{T}$ forms a commutative ring with unity. If either $\alpha_{\widehat{1}}$ or $\alpha_{\widehat{2}} = 0$, then α is a zero divisor and the set of such numbers makes up the null cone $\mathcal{NC}$. If both $\alpha_{\widehat{1}}$ and $\alpha_{\widehat{2}}$ are real, α is called a hyperbolic number.

A bicomplex function f of n bicomplex variables μ_i is defined as an n-tuple infinite positive-integer power series with bicomplex coefficients. This entails that we can write

$$f(\vec{\mu}) = f_{\widehat{1}}(\vec{\mu}_{\widehat{1}})\,\mathbf{e_1} + f_{\widehat{2}}(\vec{\mu}_{\widehat{2}})\,\mathbf{e_2}.$$

Equipped with an appropriate scalar product and its corresponding induced $\mathbb{T}$-norm, the square integrable bicomplex functions of n real variables form a bicomplex Hilbert space.

Definitions and axioms

We introduce seven linear bicomplex operators X_i, P_j and H related by

$$H = \frac{1}{2\mu}P^2 - \frac{Ze^2}{R}, \qquad \text{with} \qquad \mu, e^2, Z \in \mathbb{R},$$

which act in a $\mathbb{T}$-module $\mathcal{M}$. We say that $X_i := X_{i1}\mathbf{e_1} + X_{i2}\mathbf{e_2}$ belongs to the null cone if either X_{i1} or $X_{i2} = 0$. Similarly, $|\psi\rangle \in \mathcal{NC}$ if and only if $|\psi\rangle_{\mathbf{1}}$ or $|\psi\rangle_{\mathbf{2}} = |0\rangle$. We extend the standard canonical commutation relation to $[X_i, P_j] = \mathbf{i_1}\hbar\xi\delta_{ij}I$ with $\xi = \xi_{\widehat{1}}\mathbf{e_1} + \xi_{\widehat{2}}\mathbf{e_2}$ and $0 < \xi_{\widehat{k}} \in \mathbb{R}$. We add the requirement that these operators be self-adjoint with respect to a bicomplex scalar product defined in a natural way. There are eigenkets $|\psi_E\rangle \notin \mathcal{NC}$ of H that correspond to eigenvalues $E \notin \mathcal{NC}$. Two eigenkets $|\psi_{E_i}\rangle, |\psi_{E_j}\rangle \notin \mathcal{NC}$ of H with $(E_i - E_j) \notin \mathcal{NC}$ are orthogonal, *i.e.* $\left(|\psi_{E_i}\rangle, |\psi_{E_j}\rangle\right) = 0$.

Eigenvalues of H

As in standard quantum mechanics, the bicomplex angular momentum operator is defined as $\vec{L} := \vec{R} \times \vec{P}$. The commutation relations between the hamiltonian H, the angular momentum $\vec{L}$ and the Runge-Lenz vector $\vec{A}$ given by

$$\vec{A} := \frac{1}{2\mu}\left(\vec{P} \times \vec{L} - \vec{L} \times \vec{P}\right) - Ze^2\frac{\vec{R}}{R},$$

lead to eigenvalues E_n of H given by

$$E_n = \left\{-\frac{\mu Z^2 e^4}{2\hbar^2\xi_{\widehat{1}}^2 n_{\widehat{1}}^2}\right\}\mathbf{e_1} + \left\{-\frac{\mu Z^2 e^4}{2\hbar^2\xi_{\widehat{2}}^2 n_{\widehat{2}}^2}\right\}\mathbf{e_2}.$$

In the hyperbolic representation,

$$E_n = -\frac{\mu Z^2 e^4}{4\hbar^2}\left\{\left[\xi_{\widehat{1}}^{-2}n_{\widehat{1}}^{-2} + \xi_{\widehat{2}}^{-2}n_{\widehat{2}}^{-2}\right] + \left[\xi_{\widehat{1}}^{-2}n_{\widehat{1}}^{-2} - \xi_{\widehat{2}}^{-2}n_{\widehat{2}}^{-2}\right]\mathbf{j}\right\},$$

whence follows an interesting symmetry property:

$$\mathbb{R}\mathrm{e}(E_n, \xi) = \mathbb{H}\mathrm{y}\left(E_n, \xi\sqrt{\mathbf{j}}\right).$$

Coordinate basis eigenfunctions of H

Using the coordinates basis representation, we write the hamiltonian H in spherical coordinates and transform the eigenvalue equation $H\psi_E(\vec{r}) = E\psi_E(\vec{r})$ into

$$\sum_k \left\{\frac{\hbar^2}{2\mu}\xi_{\widehat{k}}^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{r^2\hbar^2\xi_{\widehat{k}}^2}L_{\widehat{k}}^2\right] + \left[\frac{Ze^2}{r} + E_{\widehat{k}}\right]\right\}\psi_{E_{\widehat{k}}}\mathbf{e_k} = 0,$$

for $k = 1$ and 2. By a construction of two standard solutions on the idempotent basis, we obtain the hyperbolic wave functions as

$$\psi_E(\vec{r}) = u_{nl}(r)\,Y_{lm}(\theta, \phi),$$

with $Y_{lm} := Y_{l_{\widehat{1}}m_{\widehat{1}}}\mathbf{e_1} + Y_{l_{\widehat{2}}m_{\widehat{2}}}\mathbf{e_2}$ the hyperbolic spherical harmonics and u_{nl} the radial functions given in terms of the hyperbolic Laguerre polynomials as

$$u_{nl}(r) = \sum_k \mathbf{e_k}\sqrt{u^0_{n_{\widehat{k}}l_{\widehat{k}}}}\,\mathrm{e}^{-\zeta_{\widehat{k}}/2}\zeta_{\widehat{k}}^{l_{\widehat{k}}}L^{2l_{\widehat{k}}+1}_{n_{\widehat{k}}-l_{\widehat{k}}-1}\left(\zeta_{\widehat{k}}\right).$$

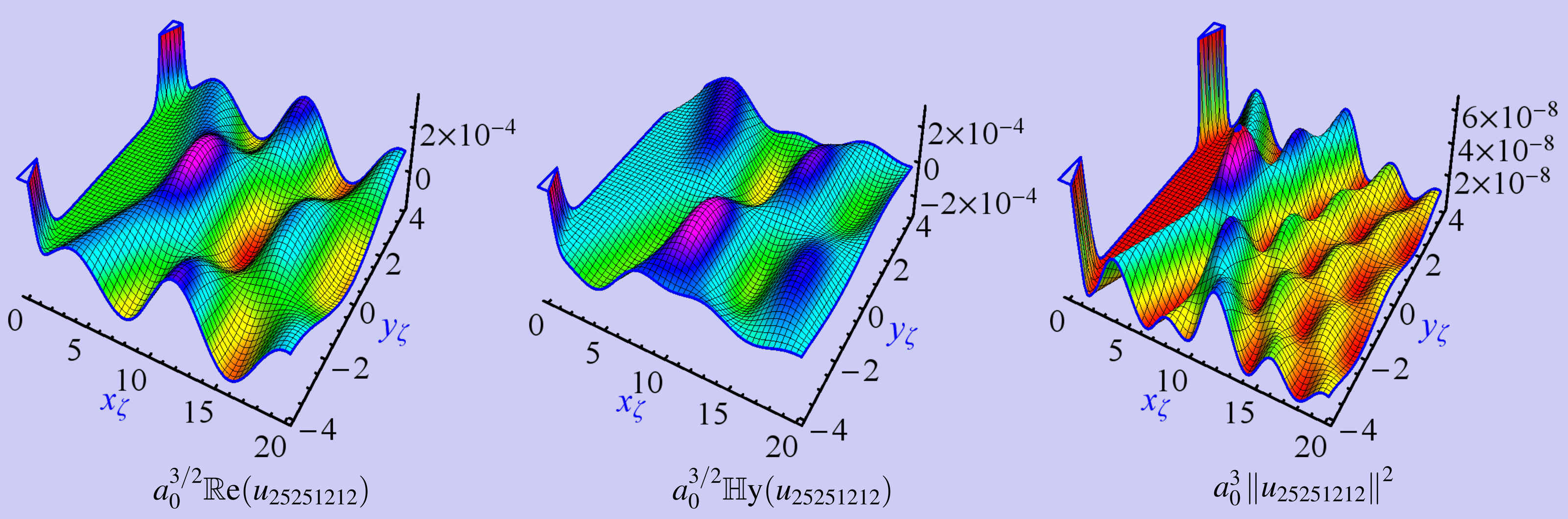

Figure. 1: Radial function in the eigenstate $\left(n_{\widehat{1}}, n_{\widehat{2}}, l_{\widehat{1}}, l_{\widehat{2}}\right) = (25, 25, 12, 12)$, with $\xi_{\widehat{1}} = \xi_{\widehat{2}} = 1$ and $\zeta := x_\zeta + y_\zeta\mathbf{j}$, where $x_\zeta = (\zeta_{\widehat{1}} + \zeta_{\widehat{2}})/2$ and $y_\zeta = (\zeta_{\widehat{1}} - \zeta_{\widehat{2}})/2$.

Conclusion

We have obtained expressions of the eigenvalues E_n and eigenfunctions ψ_{nlm} of the bicomplex hydrogen-like problem. Moreover, for $n_{\widehat{1}} = n_{\widehat{2}}$, $l_{\widehat{1}} = l_{\widehat{2}}$, $m_{\widehat{1}} = m_{\widehat{2}}$ and $\xi_{\widehat{1}} = 1 = \xi_{\widehat{2}}$, we recover the standard results as a particular case imbedded in our generalization.

References

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