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A Bicomplex Riemann Zeta Function

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The Riemann Zeta Function

The Riemann zeta function is the function of complex variable s , defined in the half-plane $Re(s) > 1$ by the convergent series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and in the «whole» complex plane $\mathbb{C}$ by analytic continuation.

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Analytic continuation of $\zeta(s)$

As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1.

A globally convergent series for the Riemann zeta function is given by :

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=0}^{\infty} \left[\frac{1}{2^{n+1}} \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{(k+1)^s} \right], s \in \mathbb{C} \setminus \{1\}.$$

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Zeros of $\zeta(s)$

The function $\zeta(s)$ has zero at the negative even integer $-2, -4, \ldots$ and one refers to them as the trivial zeros.

Riemann hypothesis :

The nontrivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

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Euler product

The connection between prime numbers and the zeta function is the celebrated **Euler product** :

$$\zeta(\sigma) = \prod_{n=1}^{\infty} \frac{1}{1 - \frac{1}{p_n^{\sigma}}}, \text{ with } \sigma \in \mathbb{R}, \sigma > 1.$$

Where $p_1, p_2, \ldots, p_n, \ldots$ is the ascending sequence of primes numbers.

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Riemann's fundamental idea is to extend Euler's formula to a complex variable. Thus he sets :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{n=1}^{\infty} \frac{1}{1-\frac{1}{p_n^s}}$$

for every complex number s with $Re(s) > 1$.

Remark :

$$p_n^s := e^{s \cdot ln(p_n)} = cos(s \cdot ln(p_n)) + i \cdot sin(s \cdot ln(p_n))$$

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Bicomplex Numbers

In 1892, in search for and development of special algebras, Corrado Segre (1860-1924) published a paper in which he treated an infinite set of algebras whose elements he called bicomplex numbers, tricomplex numbers,..., n-complex numbers.

We define **bicomplex numbers** as follows :

$$\mathbb{C}_2 := \{a + bi_1 + ci_2 + dj : {i_1}^2 = {i_2}^2 = -1,\ j^2 = 1\}$$

where $i_2 j = j i_2 = -i_1,\ i_1 j = j i_1 = -i_2,\ i_2 i_1 = i_1 i_2 = j$ and $a, b, c, d \in \mathbb{R}$.

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We remark that we can write a bicomplex number $a + bi_1 + ci_2 + dj$ as :

$$\boxed{(a + bi_1) + (c + di_1)i_2 = z_1 + z_2 i_2}$$

where $z_1, z_2 \in \mathbb{C}_1 := \{x + yi_1 : {i_1}^2 = -1\}$. Thus, $\mathbb{C}_2$ can be viewed as the *complexification* of $\mathbb{C}_1$ and a bicomplex number can be seen as an element of $\mathbb{C}^2$. Moreover, $\mathbb{C}_2$ is a **commutative unitary ring** with the following characterization for the noninvertible elements.

Let $w = z_1 + z_2 i_2 \in \mathbb{C}_2$. Then w is **noninvertible** if and only if :

$$ {z_1}^2 + {z_2}^2 = 0. $$

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Bicomplex Analysis

It is also possible to define **differentiability** of a function at a point of $\mathbb{C}_2$:

Definition 1 *Let U be an open set of $\mathbb{C}_2$ and $w_0 \in U$. Then, $f : U \subseteq \mathbb{C}_2 \longrightarrow \mathbb{C}_2$ is said to be $\mathbb{C}_2$-differentiable at w_0 with derivative equal to $f'(w_0) \in \mathbb{C}_2$ if*

$$ \lim_{\substack{w \to w_0 \\ (w - w_0 \ inv.)}} \frac{f(w) - f(w_0)}{w - w_0} = f'(w_0). $$

We will also say that the function f is $\mathbb{C}_2$-holomorphic on an open set U iff f is $\mathbb{C}_2$-differentiable at each point of U.

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As we saw, a bicomplex number can be seen as an element of $\mathbb{C}^2$, so a function $f(z_1 + z_2 i_2) = f_1(z_1, z_2) + f_2(z_1, z_2) i_2$ of $\mathbb{C}_2$ can be seen as a mapping $f(z_1, z_2) = (f_1(z_1, z_2), f_2(z_1, z_2))$ of $\mathbb{C}^2$. Here we have a characterization of such mappings :

Theorem 1 *Let U be an open set and $f : U \subseteq \mathbb{C}_2 \longrightarrow \mathbb{C}_2$. Let also $f(z_1 + z_2 i_2) = f_1(z_1, z_2) + f_2(z_1, z_2) i_2$. Then f is $\mathbb{C}_2$-holomorphic on U iff :*

$$f_1 \textit{ and } f_2 \textit{ are holomorphic in } z_1 \textit{ and } z_2$$

and,

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \textit{ and } \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \textit{ on } U.$$

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Now, it is natural to define for $\mathbb{C}^2$ the following class of mappings :

Definition 2 *The class of $\mathbb{T}$-holomorphic mappings on a open set $U \subseteq \mathbb{C}^2$ is defined as follows :*

$$TH(U) := \{ f : U \subseteq \mathbb{C}^2 \longrightarrow \mathbb{C}^2 | f \in H(U) \textit{ and}$$

$$\tfrac{\partial f_1}{\partial z_1} = \tfrac{\partial f_2}{\partial z_2}, \tfrac{\partial f_2}{\partial z_1} = -\tfrac{\partial f_1}{\partial z_2} \textit{ on } U \}.$$

It is the subclass of holomorphic mappings of $\mathbb{C}^2$ satisfying the *complexified Cauchy-Riemann* equations.

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The idempotent basis

We remark that $f \in TH(U)$ iff f is $\mathbb{C}_2$-holomorphic on U. It is also important to know that every bicomplex number $z_1 + z_2 i_2$ has the following unique idempotent representation :

$$z_1 + z_2 i_2 = (z_1 - z_2 i_1)e_1 + (z_1 + z_2 i_1)e_2$$

where $e_1 = \frac{1+j}{2}$ and $e_2 = \frac{1-j}{2}$.

This representation is very useful because : addition, multiplication and division can be done term-by-term. Also, an element will be noninvertible iff $z_1 - z_2 i_1 = 0$ or $z_1 + z_2 i_1 = 0$.

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The notion of **holomorphicity** can also be seen with this kind of notation. For this we need the following definition :

Definition 3 *We say that $X \subseteq \mathbb{C}_2$ is a $\mathbb{C}_2$-cartesian set determined by X_1 and X_2 if*

$$X = X_1 \times_e X_2 := \{z_1 + z_2 i_2 \in \mathbb{C}_2 : z_1 + z_2 i_2 =$$

$$w_1 e_1 + w_2 e_2, (w_1, w_2) \in X_1 \times X_2\}.$$

Remark :

If X_1 and X_2 are domains of $\mathbb{C}_1$ then $X_1 \times_e X_2$ is also a domain of $\mathbb{C}_2$.

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Now, it is possible to state the following striking theorems :

Theorem 2 *If* $f_{e1} : X_1 \longrightarrow \mathbb{C}_1$ *and* $f_{e2} : X_2 \longrightarrow \mathbb{C}_1$ *are holomorphic functions of* $\mathbb{C}_1$ *on the domains* X_1 *and* X_2 *respectively, then the function* $f : X_1 \times_e X_2 \longrightarrow \mathbb{C}_2$ *defined as*

$$f(z_1 + z_2 i_2) = f_{e1}(z_1 - z_2 i_1)e_1 + f_{e2}(z_1 + z_2 i_1)e_2,$$

$\forall\, z_1 + z_2 i_2 \in X_1 \times_e X_2$ *is* ***«$\mathbb{T}$-holomorphic»*** *on the domain* $X_1 \times_e X_2$.

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The Bicomplex Riemann Zeta Function

Let $n \in \mathbb{N}\backslash\{0\}$ and $w = z_1 + z_2 i_2 \in \mathbb{C}_2$. We define

$$n^w := e^{w \cdot ln(n)}$$

where

$$e^{z_1 + z_2 i_2} := e^{z_1} \cdot e^{z_2 i_2} \text{ and } e^{z_2 i_2} := cos(z_2) + i_2 sin(z_2).$$

Hence,

$$\boxed{n^{z_1 + z_2 i_2} = e^{z_1 \cdot ln(n)} \cdot [cos(z_2 \cdot ln(n)) + i_2 sin(z_2 \cdot ln(n))]}$$

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Remarks :

- $e^{w_1+w_2} = e^{w_1} \cdot e^{w_2}\ \forall w_1, w_2 \in \mathbb{C}_2$.
- e^w is invertible $\forall w \in \mathbb{C}_2$.
- $e^{z_1+z_2 i_2} = (e^{z_1-z_2 i_1})e_1 + (e^{z_1+z_2 i_1})e_2\ \forall z_1 + z_2 i_2 \in \mathbb{C}_2$.

Definition 4 *Let $w = z_1 + z_2 i_2 \in \mathbb{C}_2$ with $Re(z_1) > 1$ and $|Im(z_2)| < Re(z_1) - 1$. We define a **bicomplex Riemann zeta function** $\zeta(w)$ by the following convergent series :*

$$\zeta(w) = \sum_{n=1}^{\infty} \frac{1}{n^w}.$$

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The last definition can be well justified by the following theorem :

Theorem 3 *Let $w = z_1 + z_2 i_2 \in \mathbb{C}_2$ with $Re(z_1 - z_2 i_1) > 1$ and $Re(z_1 + z_2 i_1) > 1$. Then*

$$\zeta(w) = \left[\sum_{n=1}^{\infty} \frac{1}{n^{z_1 - z_2 i_1}}\right] e_1 + \left[\sum_{n=1}^{\infty} \frac{1}{n^{z_1 + z_2 i_1}}\right] e_2.$$

Moreover,

$$\{w \in \mathbb{C}_2 \mid Re(z_1 - z_2 i_1) > 1 \text{ and } Re(z_1 + z_2 i_1) > 1\}$$

$$= \{w \in \mathbb{C}_2 \mid Re(z_1) > 1 \text{ and } |Im(z_2)| < Re(z_1) - 1\}.$$

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We will now determine the whole domain of existence of our bicomplex Riemann zeta function. In fact, if $\mathcal{O}_2$ is the set of noninvertible elements in $\mathbb{C}_2$, we extend $\zeta(w)$ as follow :

$$\boxed{\zeta(w) := \zeta(z_1 - z_2 i_1)e_1 + \zeta(z_1 + z_2 i_1)e_2}$$

on the set $\mathbb{C}_2 \backslash \{1 + \mathcal{O}_2\}$.

Remarks :

- $w \in 1 + \mathcal{O}_2 \Leftrightarrow z_1 - z_2 i_1 = 1$ or $z_1 + z_2 i_1 = 1$.

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- The set $\mathbb{C}_2 \backslash \{1 + \mathcal{O}_2\}$ is open and connected in $\mathbb{C}^2$.
- By Theorem 2, $\zeta(w)$ is $\mathbb{T}$-holomorphic on $\mathbb{C}_2 \backslash \{1 + \mathcal{O}_2\}$.
- By the identity theorem of $\mathbb{C}^2$, our analytic continuation of $\zeta(w)$ is *unique*.
- Let $w_0 \in 1 + \mathcal{O}_2$ then

$$\lim_{\substack{w \to w_0 \\ (w \notin 1 + \mathcal{O}_2)}} |\zeta(w)| = \infty.$$

Hence, the domain $\mathbb{C}_2 \backslash \{1 + \mathcal{O}_2\}$ is the *best possible*.

The Zeros of $\zeta(w)$

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Let $w = z_1 + z_2 i_2 \in \mathbb{C}_2 \backslash \{1 + \mathcal{O}_2\}$. Then,

$$\boxed{\zeta(w) = 0 \Longleftrightarrow \zeta(z_1 - z_2 i_1) = 0 \text{ and } \zeta(z_1 + z_2 i_1) = 0}$$

Hence, from the trivial zeros of $\zeta(s)$ we can obtain trivial zeros for $\zeta(w)$. More specifically, the set $z_1 + z_2 i_2 \in \mathbb{C}_2$ such that

$$\boxed{z_1 + z_2 i_2 = (-n_1 - n_2) + (-n_1 + n_2)j}$$

where $n_1, n_2 \in \mathbb{N}\backslash\{0\}$ will be defined as the set of the **trivial zeros** for the bicomplex Riemann zeta function.

The Bicomplex Riemann Hypothesis

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Moreover, we can establish a bicomplex Riemann hypothesis for $\zeta(w)$ **equivalent** to the Riemann hypothesis for $\zeta(s)$:

Conjecture 1 *Let $w = z_1 + z_2 i_2 \in \mathbb{C}_2 \backslash \{1 + \mathcal{O}_2\}$. If w is a nontrivial zeros of the bicomplex Riemann zeta function then :*

$$(Re(z_1), Im(z_2)) = (\frac{1}{2}, 0)$$

$$or \quad (Re(z_1), Im(z_2)) = (\tfrac{1}{4} - n, \pm(\tfrac{1}{4} + n))$$

$$where\ n \in \mathbb{N}\backslash\{0\}.$$

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The Bicomplex Infinite Products

In the complex plane, an infinite product is said to converge if and only if at most a finite number of the factors are zero, and if the partial products formed by the nonvanishing factors tend to a finite limit which is different from zero. In the bicomplex case we have to pay attention to the divisors of zero.

Definition 5 *A bicomplex infinite product is said to converge if and only if at most a finite number of the factors are noninvertible, and if the partial products formed by the invertible factors tend to a finite limit which is invertible.*

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Theorem 4 *Let* $w_n = z_{1,n} + z_{2,n}i_2 \in \mathbb{C}_2 \backslash \mathcal{O}_2$ *be a sequence of invertible bicomplex numbers. Then,* $\prod_{n=1}^{\infty} w_n$ *converges if and only if*

$$\prod_{n=1}^{\infty}(z_{1,n} - z_{2,n}i_1) \text{ and } \prod_{n=1}^{\infty}(z_{1,n} + z_{2,n}i_1) \text{ converge.}$$

Moreover, in the case of convergence we obtain :

$$\prod_{n=1}^{\infty} w_n = \prod_{n=1}^{\infty}(z_{1,n} - z_{2,n}i_1)e_1 + \prod_{n=1}^{\infty}(z_{1,n} + z_{2,n}i_1)e_2.$$

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The Bicomplex Euler Product

Using the last theorem, we are able to establish a bicomplex Euler product :

Theorem 5 *Let* $w = z_1 + z_2 i_2 \in \mathbb{C}_2$ *with* $Re(z_1) > 1$ *and* $|Im(z_2)| < Re(z_1) - 1$*. Then :*

$$\zeta(w) = \sum_{n=1}^{\infty} \frac{1}{n^w} = \prod_{n=1}^{\infty} \frac{1}{1 - \frac{1}{p_n^w}}.$$

Where $p_1, p_2, \ldots, p_n, \ldots$ *is the ascending sequence of primes numbers.*