Slide 1

Bicomplex Shrödinger Equation

(in collaboration with Dominic Rochon*)

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Slide 2

Introduction to bicomplex numbers

The bicomplex numbers are defined as

$$\mathbb{T} := \{w_0 + w_1\mathbf{i_1} + w_2\mathbf{i_2} + w_3\mathbf{j} \mid w_0, w_1, w_2, w_3 \in \mathbb{R}\}.$$

The product of imaginary units are given by $\mathbf{i_1^2} = \mathbf{i_2^2} = -1$, $\mathbf{j}^2 = 1$and

$$\begin{aligned} \mathbf{i_1 i_2} &= \mathbf{i_2 i_1} &= \mathbf{j}, \\ \mathbf{i_1 j} &= \mathbf{j i_1} &= -\mathbf{i_2}, \\ \mathbf{i_2 j} &= \mathbf{j i_2} &= -\mathbf{i_1}. \end{aligned}$$

The bicomplex numbers are *commutative*.

We define the following two subsets $\mathbb{C}(\mathbf{i_k}) \subset \mathbb{T}$ for $k = 1, 2$ by

$$\mathbb{C}(\mathbf{i_k}) := \{x + y\mathbf{i_k} \mid \mathbf{i_k^2} = -1 \text{ and } x, y \in \mathbb{R}\}.$$

Slide 3

Conjugates for bicomplex numbers

Let $w \in \mathbb{T}$ and $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ such that $w = z_1 + z_2\mathbf{i_2}$. Then we define the three conjugations as:

$$\begin{aligned} w^{\dagger_1} &= (z_1 + z_2\mathbf{i_2})^{\dagger_1} := \overline{z}_1 + \overline{z}_2\mathbf{i_2}, \\ w^{\dagger_2} &= (z_1 + z_2\mathbf{i_2})^{\dagger_2} := z_1 - z_2\mathbf{i_2}, \\ w^{\dagger_3} &= (z_1 + z_2\mathbf{i_2})^{\dagger_3} := \overline{z}_1 - \overline{z}_2\mathbf{i_2}, \end{aligned}$$

where $\overline{z}_k$ is the standard complex conjugate of complex numbers $z_k \in \mathbb{C}(\mathbf{i_1})$. Hence for $w = z_1 + z_2\mathbf{i_2} = w_0 + w_1\mathbf{i_1} + w_2\mathbf{i_2} + w_3\mathbf{j}$ the conjugations of type 1,2 or 3 of w have, respectively, the signatures $(+-+-)$, $(++--)$ and $(+--+)$.

Slide 4

Conjugates for bicomplex numbers

The composition of conjugates is given by the *Klein group*:

$\circ$	$\dagger_0$	$\dagger_1$	$\dagger_2$	$\dagger_3$
$\dagger_0$	$\dagger_0$	$\dagger_1$	$\dagger_2$	$\dagger_3$
$\dagger_1$	$\dagger_1$	$\dagger_0$	$\dagger_3$	$\dagger_2$
$\dagger_2$	$\dagger_2$	$\dagger_3$	$\dagger_0$	$\dagger_1$
$\dagger_3$	$\dagger_3$	$\dagger_2$	$\dagger_1$	$\dagger_0$

where $w^{\dagger_0} := w \; \forall w \in \mathbb{T}$.

The three kind of conjugations all have the standard properties of conjugations.

Slide 5

The bicomplex moduli

$$|w|^2_{\mathbf{i_1}} := w \cdot w^{\dagger_2} = z_1^2 + z_2^2 \in \mathbb{C}(\mathbf{i_1}),$$

$$|w|^2_{\mathbf{i_2}} := w \cdot w^{\dagger_1} = \left(|z_1|^2 - |z_2|^2\right) + 2\mathrm{Re}(z_1\overline{z}_2)\mathbf{i_2} \in \mathbb{C}(\mathbf{i_2}),$$

$$|w|^2_{\mathbf{j}} := w \cdot w^{\dagger_3} = \left(|z_1|^2 + |z_2|^2\right) - 2\mathrm{Im}(z_1\overline{z}_2)\mathbf{j} \in \mathbb{D},$$

It is easy to verify that the inverse of w is given by

$$w^{-1} = \frac{w^{\dagger_2}}{|w|^2_{\mathbf{i_1}}}.$$

The set $\mathcal{NC}$ of zero divisors of $\mathbb{T}$, called the *null-cone*, is given by

$$\mathcal{NC} = \{z(\mathbf{i_1} \pm \mathbf{i_2}) |\ z \in \mathbb{C}(\mathbf{i_1})\}.$$

Slide 6

Hyperpolar coordinates

For $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ and $w = z_1 + z_2\mathbf{i_2} \in \mathbb{T}$, we have

$$\mathrm{e}^w = \mathrm{e}^{z_1 + z_2\mathbf{i_2}} = \mathrm{e}^{z_1}\mathrm{e}^{z_2\mathbf{i_2}} = \mathrm{e}^{z_1}(\cos z_2 + \mathbf{i_2}\sin z_2),$$

In particular for

- $z_1 = \ln r \in \mathbb{R}$ and $z_2 = \theta \in \mathbb{R}$, we obtain the *standard complex polar coordinates*;
- $z_1 = \ln \rho \in \mathbb{R}$ and $z_2 = \phi\mathbf{i_1}$ ($\phi \in \mathbb{R}$)

$$\begin{aligned} \mathrm{e}^{z_1+z_2\mathbf{i_2}} = \rho\mathrm{e}^{\phi\mathbf{j}} &= \rho\left[\cos(\phi\mathbf{i_1}) + \mathbf{i_2}\sin(\phi\mathbf{i_1})\right] \\ &= \rho\left[\cosh\phi + \mathbf{i_2}\mathbf{i_1}\sinh\phi\right] \\ &= \rho\left[\cosh\phi + \mathbf{j}\sinh\phi\right], \end{aligned}$$

the *hyperbolic polar coordinates*.

Slide 7

Bicomplex Schrödinger equation

Let us now consider an analog of the one-dimensional standard Schrödinger's equation over the bicomplex space functions:

$$\mathbf{i_1}\hbar\,\partial_t\psi(x,t) + \frac{\hbar^2}{2m}\,\partial_x^2\psi(x,t) - V(x,t)\psi(x,t) = 0$$

where

$$\psi : \mathbb{R}^2 \to \mathbb{T} \text{ and } V : \mathbb{R}^2 \to \mathbb{R}.$$

We express the wave function $\psi(x,t)$ into the *hyperpolar* coordinates as

$$\psi(x,t) = \mathrm{e}^{\alpha+\beta\mathbf{i_1}+\gamma\mathbf{i_2}+\delta\mathbf{j}},$$

where α, β, γ and δ are real functions going from $\mathbb{R}^2 \to \mathbb{R}$.

Slide 8

System of real differential equations

The system of four differential equations in terms of the four real functions α, β, γ and δ is given by:

$$\begin{aligned}
-\hbar\,\partial_t\beta \quad &+ \quad \tfrac{\hbar^2}{2m}\left[\partial_x^2\alpha + (\partial_x\alpha)^2 - (\partial_x\beta)^2 - (\partial_x\gamma)^2 + (\partial_x\delta)^2\right] - V = 0,\\
\partial_t\alpha \quad &+ \quad \tfrac{\hbar}{2m}\left[\partial_x^2\beta + 2(\partial_x\alpha\,\partial_x\beta - \partial_x\gamma\,\partial_x\delta)\right] = 0,\\
-\partial_t\delta \quad &+ \quad \tfrac{\hbar}{2m}\left[\partial_x^2\gamma + 2(\partial_x\alpha\partial_x\gamma - \partial_x\beta\partial_x\delta)\right] = 0,\\
\partial_t\gamma \quad &+ \quad \tfrac{\hbar}{2m}\left[\partial_x^2\delta + 2(\partial_x\alpha\,\partial_x\delta + \partial_x\beta\,\partial_x\gamma)\right] = 0.
\end{aligned} \qquad (\clubsuit)$$

The system ($\clubsuit$) posses a *4*-dimensional discrete group, leaving the solution set of the system invariant.

Slide 9

Discrete symmetry group

These discrete symmetry group is given by

$$\hat{P}_0 = Id. \qquad \hat{P}_1 = \begin{cases} \gamma \rightarrow -\gamma \\ \delta \rightarrow -\delta \end{cases}$$

$$\hat{P}_2 = \begin{cases} \gamma \rightarrow -\delta\mathbf{i_2} \\ \delta \rightarrow \gamma\mathbf{i_2} \end{cases} \qquad \hat{P}_3 = \begin{cases} \gamma \rightarrow \delta\mathbf{i_2} \\ \delta \rightarrow -\gamma\mathbf{i_2} \end{cases}$$

- Note that α, β are not transformed under these symmetries.
- The group of symmetries is the *Klein* group (the smallest non-cyclic).

Slide 10

Bicomplex continuity equation

$$1 \begin{cases} \partial_t(\psi\psi^{\dagger_1}) + \nabla\cdot\mathbf{J}_1(\psi) = 0, \\ J_1(\psi) = \frac{\hbar}{2m\mathbf{i_1}}(\psi^{\dagger_1}\partial_x\psi - \psi\partial_x\psi^{\dagger_1}) = \frac{\hbar}{m}\mathrm{e}^{2(\alpha+\gamma\mathbf{i_2})}\partial_x(\beta+\delta\mathbf{i_2}) \end{cases}$$

$$2 \begin{cases} \partial_t(\psi\psi^{\dagger_3}) + \nabla\cdot\mathbf{J}_2(\psi) = 0, \\ J_2(\psi) = \frac{\hbar}{2m\mathbf{i_1}}(\psi^{\dagger_3}\partial_x\psi - \psi\partial_x\psi^{\dagger_3}) = \frac{\hbar}{m}\mathrm{e}^{2(\alpha+\delta\mathbf{j})}\partial_x(\beta-\gamma\mathbf{j}) \end{cases}$$

$$3 \begin{cases} \partial_t(\psi^{\dagger_2}\psi^{\dagger_1}) + \nabla\cdot\mathbf{J}_3(\psi) = 0, \\ J_3(\psi) = \frac{\hbar}{2m\mathbf{i_1}}(\psi^{\dagger_1}\partial_x\psi^{\dagger_2} - \psi^{\dagger_2}\partial_x\psi^{\dagger_1}) = \frac{\hbar}{m}\mathrm{e}^{2(\alpha-\delta\mathbf{j})}\partial_x(\beta+\gamma\mathbf{j}) \end{cases}$$

$$4 \begin{cases} \partial_t(\psi^{\dagger_2}\psi^{\dagger_3}) + \nabla\cdot\mathbf{J}_4(\psi) = 0, \\ J_4(\psi) = \frac{\hbar}{2m\mathbf{i_1}}(\psi^{\dagger_3}\partial_x\psi^{\dagger_2} - \psi^{\dagger_2}\partial_x\psi^{\dagger_3}) = \frac{\hbar}{m}\mathrm{e}^{2(\alpha-\gamma\mathbf{i_2})}\partial_x(\beta-\delta\mathbf{i_2}) \end{cases}$$

Slide 11

Symmetries of the continuity equations

Under the discrete symmetries a solution $\psi(x,t) = \mathrm{e}^{\alpha+\beta \mathbf{i_1}+\gamma \mathbf{i_2}+\delta \mathbf{j}}$ of (♣) is transformed into:

$$\hat{P}_1(\psi) = \psi^{\dagger_2},\ \hat{P}_2(\psi) = \psi_+,\ \hat{P}_3(\psi) = \psi_-,$$

where the functions ψ_+ and ψ_- are functions in the $\mathbb{C}(\mathbf{i_1})$-space given by

$$\psi_\pm = \mathrm{e}^{(\alpha\pm\delta)+(\beta\mp\gamma)\mathbf{i_1}}.$$

Consider now the discrete symmetries $\hat{P}_1, \hat{P}_2$ and $\hat{P}_3$ on the continuity equations:

- $\hat{P}_1$: The continuity equations $1 \to 4$ and $2 \to 3$;

Slide 12

Symmetries of the continuity equations

- $\hat{P}_2$: The continuity equations 1 and 2 are both transformed into:

$$\partial_t(\psi_+\overline{\psi}_+) + \nabla \cdot \mathbf{J}(\psi_+) = 0,$$

$$J(\psi_+) = \frac{\hbar}{2m\mathbf{i_1}}(\overline{\psi}_+\partial_x\psi_+ - \psi_+\partial_x\overline{\psi}_+) = \frac{\hbar}{m}\mathrm{e}^{2(\alpha+\delta)}\partial_x(\beta-\gamma).$$

- $\hat{P}_3$: The continuity equations 1 and 2 are both transformed into:

$$\partial_t(\psi_-\overline{\psi}_-) + \nabla \cdot \mathbf{J}(\psi_-) = 0,$$

$$J(\psi_-) = \frac{\hbar}{2m\mathbf{i_1}}(\overline{\psi}_-\partial_x\psi_- - \psi_-\partial_x\overline{\psi}_-) = \frac{\hbar}{m}\mathrm{e}^{2(\alpha-\delta)}\partial_x(\beta+\gamma).$$

Slide 13

Definitions of *real* moduli

Motivation: Obtain some bicomplex *Born formulas*.

Let $s, t \in \mathbb{T}$, we define $|\cdot|_k$, $k = 1, 2, 3$ as

1. $|\cdot|_{\mathbf{1}} := \big||\cdot|_{\mathbf{i_1}}\big|$,

   For $w = z_1 + z_2\mathbf{i_2}$ with $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ we have

   $$|w|_{\mathbf{1}} = |z_1^2 + z_2^2|^{1/2} = \sqrt[4]{ww^{\dagger_1}w^{\dagger_2}w^{\dagger_3}}.$$

   This modulus has the following properties:

   (a) $|\cdot|_{\mathbf{1}} : \mathbb{T} \to \mathbb{R}$;

   (b) $|s|_{\mathbf{1}} \geq 0$ with $|s|_{\mathbf{1}} = 0$ iff $s \in \mathcal{NC}$;

   (c) $|s \cdot t|_{\mathbf{1}} = |s|_{\mathbf{1}} \cdot |t|_{\mathbf{1}}$.

Slide 14

Definitions of *real* moduli

2. $|\cdot|_{\mathbf{2}} := \big||\cdot|_{\mathbf{i_2}}\big|$,

   This modulus has the same properties as $|\cdot|_{\mathbf{1}}$. Indeed we can rewrite $|w|_{\mathbf{2}}$ as $|w|_{\mathbf{2}} = |z_1^2 + z_2^2|^{1/2}$, where $w = z_1 + z_2\mathbf{i_1}$ with $z_1, z_2 \in \mathbb{C}(\mathbf{i_2})$.

3. $|\cdot|_{\mathbf{3}} := \big||\cdot|_{\mathbf{j}}\big|$,

   For $w = z_1 + z_2\mathbf{i_2}$ with $z_1, z_2 \in \mathbb{C}(\mathbf{i_1})$ we have

   $$|w|_{\mathbf{3}} = |w| = \sqrt{\mathrm{Re}(|w|_{\mathbf{j}}^2)} = \sqrt{|z_1|^2 + |z_2|^2}.$$

   This modulus has the following properties:

   (a) $|\cdot|_{\mathbf{3}} : \mathbb{T} \to \mathbb{R}$ (b) $|s|_{\mathbf{3}} \geq 0$ with $|s|_{\mathbf{3}} = 0$ iff $s = 0$

   (c) $|s + t|_{\mathbf{3}} \leq |s|_{\mathbf{3}} + |t|_{\mathbf{3}}$ (d) $|s \cdot t|_{\mathbf{3}} \leq \sqrt{2}\,|s|_{\mathbf{3}} \cdot |t|_{\mathbf{3}}$.

Slide 15

Invariance of the real moduli

Under the bicomplex Born formulas, the wave function $\psi(x,t) = \mathrm{e}^{\alpha+\beta\mathbf{i_1}+\gamma\mathbf{i_2}+\delta\mathbf{j}}$ becomes

$$|\psi|^2_{\mathbf{1}} = |\psi|^2_{\mathbf{2}} = \mathrm{e}^{2\alpha} \longrightarrow \text{standard case}$$

$$|\psi|^2_{\mathbf{3}} = \mathrm{e}^{2\alpha}\cosh(2\delta) = \mathrm{e}^{2\alpha}\underbrace{\left(1+\frac{(2\delta)^2}{2!}+\frac{(2\delta)^4}{4!}+\cdots\right)}_{\text{hyperbolic perturbation}}.$$

What is the result of the bicomplex Born formulas under the discrete symmetries?

Slide 16

Invariance of the real moduli

$$|\hat{P}_1\psi|^2_{\mathbf{k}} = |\psi|^2_{\mathbf{k}} = \begin{cases} \mathrm{e}^{2\alpha} & \text{if } k=1,2 \\ \mathrm{e}^{2\alpha}\cosh(2\delta) & \text{if } k=3 \end{cases}$$

and

$$|\hat{P}_2\psi|^2_{\mathbf{k}} = \mathrm{e}^{2\delta}|\psi|^2_{\mathbf{k}} = \mathrm{e}^{2(\alpha+\delta)},$$

$$|\hat{P}_3\psi|^2_{\mathbf{k}} = \mathrm{e}^{-2\delta}|\psi|^2_{\mathbf{k}} = \mathrm{e}^{2(\alpha-\delta)}$$

for $k = 1,2,3$.

Slide 17

Born formulas for $\delta(x,t) \to 0$

Let $\mathbf{e_1} := \frac{1+\mathbf{j}}{2}$ and $\mathbf{e_2} := \frac{1-\mathbf{j}}{2}$. Then we can express $\psi(x,t)$ as

$$\psi = \mathrm{e}^{\alpha+\beta\mathbf{i_1}+\gamma\mathbf{i_2}+\delta\mathbf{j}} = \psi_+\mathbf{e_1} + \psi_-\mathbf{e_2}.$$

THEOREM. Let ψ be a complex wave function given by

$$\psi(x,t) = \mathrm{e}^{\alpha(x,t)+\beta(x,t)\mathbf{i_1}+\gamma(x,t)\mathbf{i_2}} = \psi_+(x,t)\mathbf{e_1} + \psi_-(x,t)\mathbf{e_2}.$$

Then

$$|\psi|^2 = |\psi|^2_{\mathbf{1}} = |\psi|^2_{\mathbf{2}} = |\psi|^2_{\mathbf{3}} = \sqrt{\psi\psi^{\dagger_1}\psi^{\dagger_2}\psi^{\dagger_3}} = \frac{|\psi_+|^2 + |\psi_-|^2}{2} = \mathrm{e}^{2\alpha},$$

where $|\psi|^2$ gives the standard Born's formula and is invariant under all the discrete symmetries of the bicomplex Schrödinger equation.

Slide 18

Conclusion

- The fact that hyperbolic angle of the exponential is zero in ψ , i.e. we have to consider $\delta(x,t) = 0$ in the THEOREM, do not mean that *hyperbolic part* of the wave function do not play any role. Indeed, we can rewrite ψ as

$$\begin{aligned}\psi(x,t) \quad &= \quad \mathrm{e}^{\alpha}\left(\cos\beta\cos\gamma + \mathbf{i_1}\sin\beta\cos\gamma + \mathbf{i_2}\cos\beta\sin\gamma\right. \\ &\quad \left. + \,\mathbf{j}\sin\beta\sin\gamma\right).\end{aligned}$$

  Hence, the wave function considered in the THEOREM is really a *bicomplex function*.

- Under some discrete symmetries of the system (♣) of the bicomplex Schrödinger equation, the bicomplex continuity equations can be transformed into *real continuity equations*.
- Generalization of the Born's formula for a class of wave funct.