Zakharov-Shabat system and hyperbolic pseudoanalytic function theory

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Outline

1. Introduction to hyperbolic analysis
2. Hyperbolic pseudoanalytic function theory
3. Zakharov-Shabat system and pseudoanalytic function theory

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Basic results on hyperbolic numbers

Let us consider the set of hyperbolic numbers:

$$\mathbb{D} := \left\{ z = x + t\mathrm{j} \ : \ \mathrm{j}^2 = 1, \ x, t \in \mathbb{R} \right\} \cong \mathrm{Cl}_{\mathbb{R}}(0,1).$$

We define $\bar{z} := x - t\mathrm{j}$ and $|z|^2 := z\bar{z} = x^2 - t^2 \in \mathbb{R}$.
We can verify that the inverse of z whenever exists is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2}.$$

The set $\mathcal{NC}$ of zero divisors is given by

$$\mathcal{NC} = \left\{ x + t\mathrm{j} \ : \ |x| = |t| \right\}.$$

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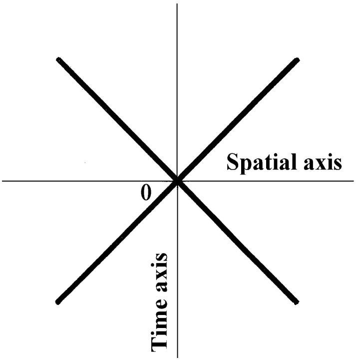
Null-cone of hyperbolic numbers



Hyperbolic analysis

Theorem

Let U be an open set and $f : U \subseteq \mathbb{D} \longrightarrow \mathbb{D}$ such that $f \in C^1(U)$. Let also $f(x + t\mathrm{j}) = f_1(x, t) + f_2(x, t)\mathrm{j}$. Then f is $\mathbb{D}$-holomorphic on U iff

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial t} \quad \text{and} \quad \frac{\partial f_2}{\partial x} = \frac{\partial f_1}{\partial t}. \tag{1}$$

Moreover $f' = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial x}\mathrm{j}$ and $f'(z)$ is invertible iff $\det \mathcal{J}_f(z) \neq 0$.

The system of linear PDEs (1) is the "*Hyperbolic Cauchy-Riemann equations*".

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Elementary hyperbolic derivative

We define $\partial_z = \frac{1}{2}(\partial_x + \mathrm{j}\partial_t)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x - \mathrm{j}\partial_t)$.
For a function $f(z) = u(x,t) + v(x,t)\mathrm{j}$ we note that

$$f_z = \frac{1}{2}\Big((u_x + v_t) + (v_x + u_t)\mathrm{j}\Big) \quad \text{and} \quad f_{\bar{z}} = \frac{1}{2}\Big((u_x - v_t) + (v_x - u_t)\mathrm{j}\Big).$$

In view of these operators,

$$f_z(z) = 0 \quad \Leftrightarrow \quad u_x = -v_t, \; v_x = -u_t$$

$$f_{\bar{z}}(z) = 0 \quad \Leftrightarrow \quad u_x = v_t, \; v_x = u_t$$

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Lemma

Let $f(z) = u(x,t) + v(x,t)\mathrm{j}$ *be a hyperbolic function where* u_x, u_t, v_x *and* v_t *exist, and are continuous in a neighborhood of* z_0. *The derivative*

$$f'(z_0) = \lim_{\substack{z \longrightarrow z_0 \\ (z - z_0 \ \ inv.)}} \frac{f(z) - f(z_0)}{z - z_0}$$

exists iff

$$f_{\bar{z}}(z_0) = 0.$$

Moreover, $f'(z_0) = f_z(z_0)$ *and* $f'(z_0)$ *is invertible iff* $\det \mathcal{J}_f(z_0) \neq 0$.

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Hyperbolic pseudoanalytic function theory

- Let $z = x + t\mathrm{j} \in \mathbb{D}$, where $x, t \in \mathbb{R}$.
- The theory is based on assigning the part played by 1 and j to two essentially arbitrary functions $F(x,y)$ and $G(x,y)$ twice continuously differentiable in some open domain $\Omega \subset \mathbb{D}$:

$$\begin{array}{lllll} \text{Standard:} & W(z) = \phi(x,y) & 1 & + \ \psi(x,y) & \mathrm{j} \\ & & \downarrow & & \downarrow \\ \text{Pseudoanalytic:} & W(z) = \phi(x,y) & F(x,y) & + \ \psi(x,y) & G(x,y). \end{array}$$

- We require that $\operatorname{Im}\left(\overline{F(z)}G(z)\right) \neq 0$ in Ω.
- Under this condition (F, G) will be called a generating pair in Ω.

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- Notice that $\mathrm{Im}(\overline{F(z)}G(z)) = \begin{vmatrix} \mathrm{Re}\{F(z)\} & \mathrm{Re}\{G(z)\} \\ \mathrm{Im}\{F(z)\} & \mathrm{Im}\{G(z)\} \end{vmatrix}$.
- From Cramer's theorem, for every z_0 in Ω we can find unique constants $\lambda_0, \mu_0 \in \mathbb{R}$ such that $w(z_0) = \lambda_0 F(z_0) + \mu_0 G(z_0)$.
- More generally we have the following result.

Theorem

Let (F, G) be generating pair in some open domain Ω. If $w(z) : \Omega \to \mathbb{D}$, then there exist **unique** *functions $\phi(z), \psi(z) : \Omega \subset \mathbb{D} \to \mathbb{R}$ such that*

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Moreover, we have the following explicit formulas for ϕ and ψ:

$$\phi(z) = \frac{\mathrm{Im}[\overline{w(z)}G(z)]}{\mathrm{Im}[\overline{F(z)}G(z)]}, \ \psi(z) = -\frac{\mathrm{Im}[\overline{w(z)}F(z)]}{\mathrm{Im}[\overline{F(z)}G(z)]}.$$

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We say that $w : \Omega \subset \mathbb{D} \to \mathbb{D}$ possesses at z_0 the (F, G)-derivative $\dot{w}(z_0)$ if the limit

$$\dot{w}(z_0) = \lim_{\substack{z \to z_0 \\ (z - z_0 \ \ inv.)}} \frac{w(z) - \lambda_0 F(z) - \mu_0 G(z)}{z - z_0}$$

exists.

The following expressions are called the "*characteristic coefficients*" of the pair (F, G):

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Equation (2) can be rewritten in the following form

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Equation (2) is called "*Vekua equation*". Any solutions of (2) are called "*(F, G)-pseudoanalytic functions*".

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Let (F,G) and (F_1,G_1) - be two generating pairs in Ω. (F_1,G_1) is called successor of (F,G) and (F,G) is called predecessor of (F_1,G_1) if

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Formal powers

Let w be an (F, G)-pseudoanalytic function. Using a generating sequence in which (F, G) is embedded we can define the higher derivatives of w by

$$w^{[0]} = w; \qquad w^{[m+1]} = \frac{\mathrm{d}_{(F_m, G_m)} w^{[m]}}{\mathrm{d}z}, \quad m = 0, 1, 2, \ldots$$

Formal powers $Z_m^{(n)}(a, z_0; z)$ with center $z_0 \in \Omega$, coefficient a and exponent n can be introduced by the following relations

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This last definition implies the following properties.

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3. The formal powers satisfy the differential relations
$$\frac{\mathrm{d}_{(F_m,G_m)} Z_m^{(n)}(a, z_0; z)}{dz} = n Z_{m+1}^{(n-1)}(a, z_0; z).$$
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$$Z_m^{(n)}(a, z_0; z) \sim a(z - z_0)^n, \quad z \to z_0$$
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Zakharov-Shabat system

1. Inverse scattering problems involving coupling mode have been investigated by many authors.
2. When the medium concerned is treated as a continuously varying one, the 1-D case is usually associated with Zakharov-Shabat (ZS) coupling-mode equations.
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Zakharov-Shabat system & hyperbolic Vekua eq.

Consider the ZS system for two modes n_1 and n_2:

$$\begin{aligned} \partial_x n_1 + \mathrm{i}kn_1 &= s(x)n_2 \\ \partial_x n_2 - \mathrm{i}kn_2 &= -s(x)n_1, \end{aligned}$$

where the functions n_1, n_2, the coupling potential $s(x)$ and the wavenumber parameter k are supposed to be complex.

This system is frequently considered as a Fourier transform of the following system

$$\begin{aligned} \partial_x n_+ + \partial_t n_+ &= s(x)n_- \\ \partial_x n_- - \partial_t n_- &= -s(x)n_+. \end{aligned}$$

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Consider the following functions

$$u = n_- + n_+, \qquad v = n_- - n_+.$$

We have

$$\begin{aligned} \partial_x u - \partial_t v &= sv \\ \partial_x v - \partial_t u &= -su. \end{aligned}$$

This system can be written in the form

$$W_{\bar{z}} = -\frac{s(x)\mathrm{j}}{2}\overline{W},$$

where $z = x + \mathrm{j}t$, $W = u + \mathrm{j}v$, $W_{\bar{z}} = \frac{1}{2}(\partial_x - \mathrm{j}\partial_t)W$.

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Zakharov-Shabat system & generating pair

We are able to construct a corresponding generating pair:

$$F(x) = \cos S(x) - \mathrm{j} \sin S(x), \qquad G(x) = \sin S(x) + \mathrm{j} \cos S(x),$$

where S is an antiderivative of s.

Notice that $\mathrm{Im}(\overline{F}G) \equiv 1$.

In order to introduce the (F, G)-derivative in the sense of Bers let us calculate the characteristic coefficients $A_{(F,G)}$, $B_{(F,G)}$:

$$F_{\bar{z}} = F_z = -\frac{s}{2} G \qquad \text{and} \qquad G_{\bar{z}} = G_z = \frac{s}{2} F.$$

Then

$$A_{(F,G)} = 0 \qquad \text{and} \qquad B_{(F,G)} = -\frac{s(x)\mathrm{j}}{2}.$$

Zakharov-Shabat system & generating pair

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$$F(x) = \cos S(x) - \mathrm{j} \sin S(x), \qquad G(x) = \sin S(x) + \mathrm{j} \cos S(x),$$

where S is an antiderivative of s.

Notice that $\mathrm{Im}(\overline{F}G) \equiv 1$.

In order to introduce the (F, G)-derivative in the sense of Bers let us calculate the characteristic coefficients $A_{(F,G)}$, $B_{(F,G)}$:

$$F_{\overline{z}} = F_z = -\frac{s}{2}G \qquad \text{and} \qquad G_{\overline{z}} = G_z = \frac{s}{2}F.$$

Then

$$A_{(F,G)} = 0 \qquad \text{and} \qquad B_{(F,G)} = -\frac{s(x)\mathrm{j}}{2}.$$

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Zakharov-Shabat system & generating pair

Thus, the (F, G)-derivative of solutions of the main Vekua has the form

$$\dot{W} = W_z + \frac{s(x)\mathrm{j}}{2}\overline{W}$$

and is a solution of the equation

$$(\dot{W})_{\overline{z}} = \frac{s(x)\mathrm{j}}{2}\overline{w}$$

for which a generating pair can be constructed as well

$$F_1(x) = \cos S(x) + \mathrm{j}\sin S(x), \qquad G_1(x) = -\sin S(x) + \mathrm{j}\cos S(x).$$

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Zakharov-Shabat system & generating sequence

The generating sequence $\{(F_m, G_m)\}$ has then the form

$$F_m = \cos S(x) + (-1)^{m+1} \mathrm{j} \sin S(x), \;\; G_m = (-1)^m \sin S(x) + \mathrm{j} \cos S(x),$$

with

$$\left(W^{[n]}\right)_{\overline{z}} = (-1)^{n+1} \frac{s(x)\mathrm{j}}{2} \overline{W^{[n]}}$$

$$\Updownarrow$$

$$W^{[n+1]} = \left(W^{[n]}\right)_z + (-1)^n \frac{s(x)\mathrm{j}}{2} \overline{W^{[n]}}.$$

That is, it is periodic with a period 2.

In this case the whole system of formal powers can be constructed explicitly and

$$F_m^* = G_m, \quad G_m^* = F_m.$$

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Thank you !

Sébastien Tremblay
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