The Generalized Mandelbrot Sets over the Tricomplex Spaces

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Introduction

In 1980, the mathematician Benoît Mandelbrot visualized for the first time on a computer many mathematical objects that he called *fractals*.

One of the most popular fractals is the Mandelbrot set, obtained from the iteration of the polynomial function $z^2 + c$. A few years later, factals images were generated from polynomials of the form $z^p + c$. They are usually called *multibrot* sets.

These fractals are two-dimensional (2D). Many mathematicians were interested in exploring these in three-dimensional space (3D).

In 2000, D. Rochon published a paper that proposed an efficient method to observe fractal objects in 3D. His method uses a particular number structure: the bicomplex numbers.

Then, V. Garant-Pelletier and D. Rochon studied the Mandelbrot set on the space of tricomplex numbers. More generally, they studied the same set in the multicomplex spaces. They generalized many results associated to the classical Mandelbrot set into these new structures.

In this poster, the results from V. Garant-Pelletier and D. Rochon's article are generalized to multibrot sets.

Questions

- What are the shapes of the fractal sets generated by polynomial functions of the form $z^p + c$ where z and c are complex numbers and p is an integer greater than or equal to 2?
- What are the properties of these sets?
- Can we generate these sets with another number structure such as the bicomplex or the tricomplex numbers?
- Is it possible to construct an algorithm to generate these fractal images?

Definitions

1. **Complex numbers:** A complex number in $\mathbb{C} \simeq \mathbb{M}(1)$ is defined as follow:

$$z = x_1 + x_2\mathbf{i_1} \tag{1}$$

where $\mathbf{i_1}^2 = -1$ and x_1, x_2 are real numbers.

2. **Bicomplex numbers:** A bicomplex number in $\mathbb{M}(2)$ is defined as a quadruplet of real numbers:

$$\zeta = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{j_1} \tag{2}$$

where $\mathbf{i_1}^2 = \mathbf{i_2}^2 = -1$, $\mathbf{j_1}^2 = 1$ and $x_i \in \mathbb{R}$.

3. **Tricomplex numbers:** A tricomplex number in $\mathbb{M}(3)$ is defined as an octuplet of real numbers:

$$\eta = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{i_3} + x_5\mathbf{i_4} + x_6\mathbf{j_1} + x_7\mathbf{j_2} + x_8\mathbf{j_3} \tag{3}$$

where $\mathbf{i_4} = \mathbf{i_1}\mathbf{i_2}\mathbf{i_3}$, $\mathbf{i_3}^2 = \mathbf{i_4}^2 = -1$ and $\mathbf{j_2}^2 = \mathbf{j_3}^2 = 1$.

Definitions (continued...)

Multibrot sets: The Multibrots are based on a simple polynomial function:

$$Q_{p,c}(z) = z^p + c \tag{4}$$

iterated many times from the starting point $z = 0$. The variable z and the fixed number c are complex numbers, bicomplex numbers or tricomplex numbers and p is an integer greater than or egal to 2. Precisely, the definition of the multibrot sets is:

$$\mathcal{M}_i^p := \left\{ c \in \mathbb{M}(i) : \left\{ Q_{p,c}^n(0) \right\}_{n=1}^{\infty} \text{ is bounded} \right\} \tag{5}$$

where $i = 1, 2$ or 3.

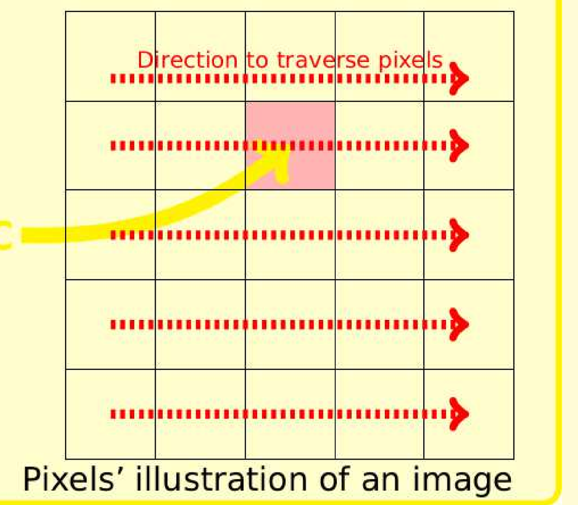
Theory and Method

To visualize the Multibrots in two or three dimensions, two important results have to be considered.

Theorem 1 *If a number c belongs to a Multibrot $\mathcal{M}_i^p$, then its modulus is less than or equal to $2^{1/(p-1)}$.*

Theorem 2 *A number c belongs to a Multibrot if, and only if, the modulus of its iterates $|Q_{p,c}^n(0)|$ do not exceed $2^{1/(p-1)}$ for any integer $n \geq 1$.*

Theorem 1 says that the multibrot sets are inside a discus (in 2D) or a ball (in 3D) with a radius $2^{1/(p-1)}$. This fact restricts the region to explore and to test if a number c belongs to a Multibrot. This test is described in the statement of Theorem 2: For each number c (e.g., in the complex plane), compute the succesive iterates $Q_{p,c}(0) = c$, $Q_{p,c}^2(0) = Q_{p,c}(Q_{p,c}(0)) = Q_{p,c}(c) = c^p + c$, etc. and verify that its modulus, at each step, is less than or equal to $2^{1/(p-1)}$.



Pixels' illustration of an image

To perform the test in the plane, we fix a domain of exploration (of complex points). These points are associated with the pixels of a given image (the pink square in the figure on the left). To make the algorithm easier, we take a squared domain where each side length is $2 \cdot 2^{1/(p-1)}$.

Then, we fix a maximum number of iterations from which we accept that a number is in a Multibrot.

If the computations mentionned above are verified up to the maximum number of iterations, then the pixel that is associated to the number c is assigned a given color.

For Multibrots in the 3D space, the technique is similar. We choose three components of the bicomplex or the tricomplex numbers and fix the remaining components to 0. Then, instead of taking a square, we use a cubic domain with side length equal to $2 \cdot 2^{1/(p-1)}$. Once more, we execute the same steps as in the 2D case, adapted to the 3D domain.

Rendering results

In 2D:

By varying the integer p, we get many fractal images. Black regions represent multibrot sets. Colors represent the divergence's speed of iterates.

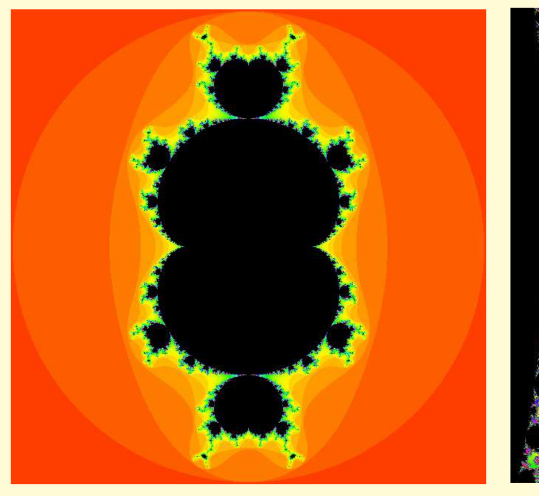


Figure 1: $\mathcal{M}_1^3$ in square $[-\sqrt[2]{2}, \sqrt[2]{2}] \times [-\sqrt[2]{2}, \sqrt[2]{2}]$

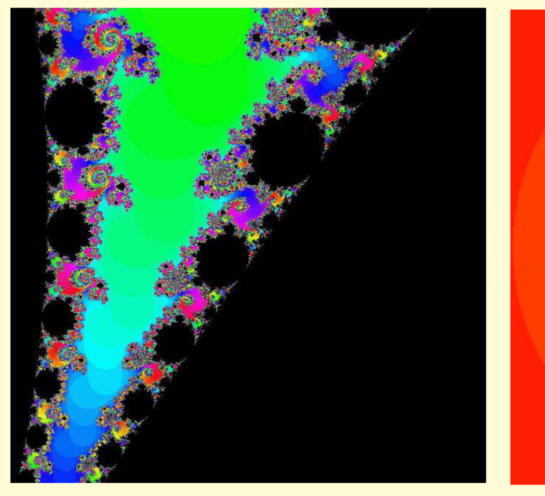


Figure 2: $\mathcal{M}_1^6$ zoomed in a region

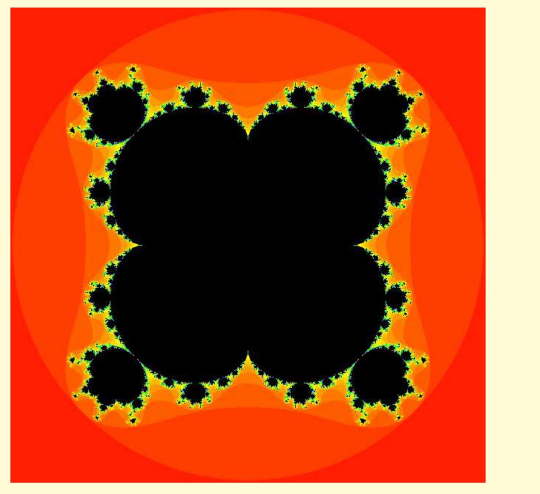


Figure 3: $\mathcal{M}_1^5$ in square $[-\sqrt[4]{2}, \sqrt[4]{2}] \times [-\sqrt[4]{2}, \sqrt[4]{2}]$

In 3D:

There are many possibilities for the 3D slices of a Multibrot. The following images give an idea of all the possibilities that are available.

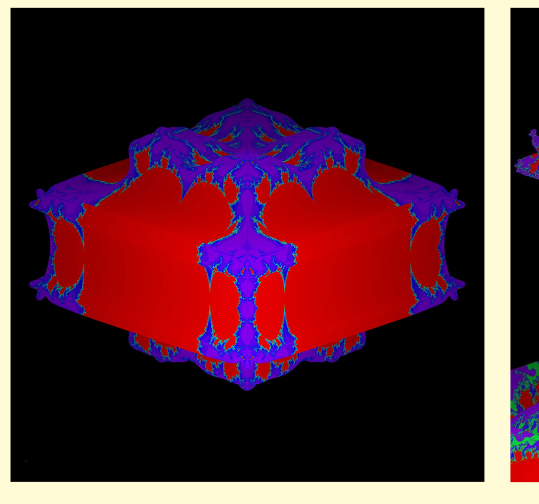


Figure 4: $\mathcal{M}_3^3$ slice $(1, \mathbf{i_1}, \mathbf{i_2})$

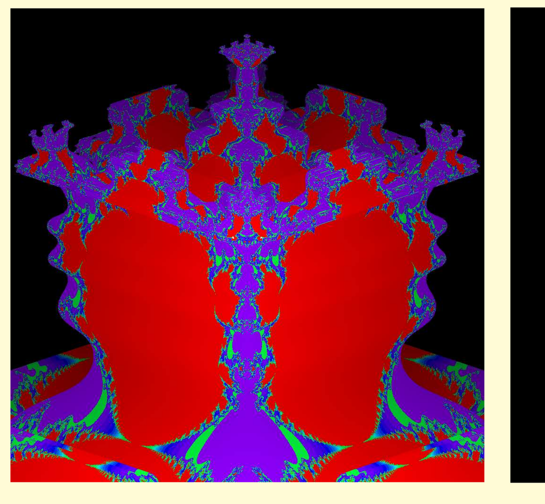


Figure 5: $\mathcal{M}_3^4$ slice $(1, \mathbf{i_1}, \mathbf{i_2})$

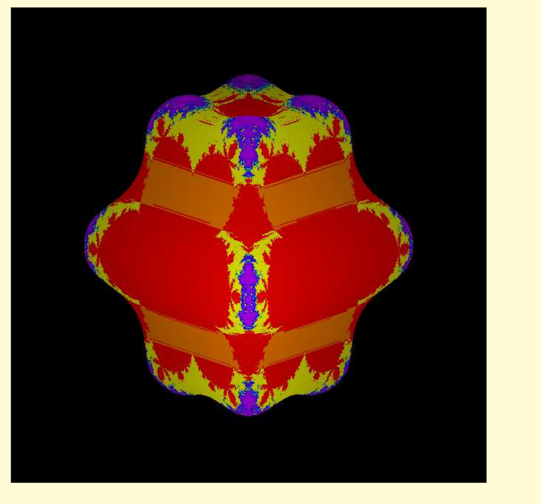


Figure 6: $\mathcal{M}_3^4$ slice $(\mathbf{i_1}, \mathbf{j_2}, \mathbf{j_3})$

Conclusion

The Generalized Multibrots have a rich fractal structure. They may be used in virtual reality or movies. Currently, two questions remain unanswered in general:

1. What is the link between the real intersection of Multibrots and their tricomplex versions in general?
2. If we can classify the 3D slices, how many classes are there in general?

References

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